ANISOTROPIC SCATTERING IN NEUTRON TRANSPORT THEORY

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ANISOTROPIC SCATTERING IN NEUTRON

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TABLE OF CONTENTS

LIST OF FIGURES

SUMMARY

The anisotropy of the neutron scattering process in the center of mass reference frame can be introduced into neutron transport theory by means of what may be called input functions. The primary purpose of this research was to derive relations between some frequently used input functions, such as the average logarithmic energy decrement, and the experimentally determined angular distributions of the neutron scattering cross sections, which have been converted to the center of mass reference frame.

In order to study the anisotropy of the scattering process, it was first necessary to examine in some detail the nonrelativistic theory of neutron scattering. The mechanics of the collision process was considered and a transformation relation between the cosines of the scattering angles, μ_o in the laboratory and μ in the center of mass reference frame, was recalled. An expansion for the function $f(\mu)d\mu$ which represents the fraction of all scattering collisions for a particular initial energy E' which result in scattering angles in the center of mass (C) system whose cosines lie between μ and $\mu + d\mu$ was given. This function was related to the differential scattering cross section in the laboratory system.

The process of averaging a function over the angular distribution of the neutron scattering cross section in the C system was examined. Expressions were derived for averaging a function with respect to $f(\mu)$ or $f(E)$.

The scattering function $\Sigma_s(E', \mu_o)$ was defined in terms of the macroscopic cross section $\Sigma_s(E', \mu_o)d\mu_o$ at an energy E' for scattering

through an angle in the laboratory (L) system whose cosine lies between μ_o and $\mu_o + d\mu_o$. A series representation of $\Sigma_s(E', \mu_o)$ in terms of Legendre polynomials in μ_o was established.

The transference function, $\Sigma_s(\vec{r}, E' \rightarrow E, \hat{\Omega}' \rightarrow \hat{\Omega})$, which represents the macroscopic cross section at position $\vec{r}$ for scattering from an initial energy E' to a final energy which lies between E and E + dE and from an initial direction characterized by a unit vector $\hat{\Omega}'$ into a unit solid angle about $\hat{\Omega}$ was introduced. A series expansion of the transference function in terms of the Legendre polynomials $P_\ell(\mu_o)$ and $P_\ell(\mu)$ was made. It was found, in obtaining this expansion, that μ_o is given by some function, say g, of the initial and final neutron energies. Expressions were supplied for $g(E', E)$ for both elastic and inelastic collisions. It appears, from the form of $g(E', E)$, that a neutron with a given initial energy which was scattered through a given angle in the L system might have two possible final energies. This situation was examined in detail.

Apart from its state of polarization, a neutron ensemble is completely described by the Boltzmann transport equation. The transport equation was derived rigorously from a statistical viewpoint. In order to facilitate approximate solutions, the spherical harmonics form of the transport equation was developed. The one dimensional spherical harmonics form was then obtained by restricting the source function and the cross sections to be functions of only one space coordinate.

A transformation relation between the flux due to an isotropic infinite plane source and that resulting from an isotropic point source was given. It was then shown that there was no loss in generality involved in considering only the one dimensional case if the source function could be built up from isotropic point sources since, for this type of source, the one dimensional results can be transformed to the three dimensional case.

As a simple application of some input functions, a P_2 approximation to the one dimensional monoenergetic transport equation was considered. A solution was obtained for the scalar flux by means of the Green's function for the problem. This solution contains the average cosine ($\overline{\mu}_o$) and the average cosine squared ($\overline{\mu_o^2}$) of the scattering angle in the L system.

A comparison was made between this P_2 approximation to the scalar flux, the ordinary diffusion theory solution, and the asymptotic form of the rigorous solution of the transport equation for an isotropic infinite plane source in a weakly absorbing medium.

A method of analysis which is often used in nuclear reactor core studies is the Fermi age theory. A further objective of this research was to develop an improved age theory which would include anisotropic scattering effects of first and second order in $P_\ell(\mu)$.

The customary age equation can be obtained from a P_1 approximation to the transport equation if the neutron scattering is assumed to be isotropic in the C system. A slightly more accurate age theory was here obtained by using a P_2 approximation to the transport equation, but still considering only S wave scattering. Next, an improved age theory was developed; first by consistently using P_1 approximations, and then extending the results to the case where a P_2 approximation to the transport equation was considered. The age theory obtained in this manner contains second order anisotropic scattering effects. It represents the optimum age theory in that the highest order approximations which will yield an age type theory were used.

In all of these age theory derivations, an effort was made to dispense with intuitive arguments and to proceed along a chain of rigor with no links missing.

The limitations of the optimum age theory were investigated. The assumptions which were employed in the development of the theory were examined for the case of an infinte medium containing an infinite plane source of monoenergetic neutrons. A validity criterion was derived for each of these assumptions.

The advantage of using the optimum age theory instead of an age theory based on a P_1 approximation to the transport equation can not be accurately assessed at the present time. An exact comparison of the results given by the two theories for a particular case would require a detailed knowledge of the energy dependence of the angular distribution of the neutron scattering cross section. The experimental cross section data which is available at this time is inadequate for such a comparison. However, the isotropic scattering limit of the optimum age theory was examined for the case of an isotropic infinite plane source of one Mev neutrons in water. A comparison was made, for this case, between the slowing down density predicted by the ordinary age theory and that given by the optimum age theory in the isotropic scattering limit.

In the course of the considerations of the neutron slowing down process involved in the development of the optimum age theory, it was found necessary to examine the effect of the anisotropy of the scattering process in the C system on the one dimensional slowing down functions $q_\ell(z, u)$. The $\ell = 0$ term represents the slowing down density at position z and lethargy u. The $\ell = 1$ term constitutes the slowing down current density in the z direction. By expanding the scattering collision density in a Taylor series, a general series expression for $q_\ell(z, u)$ was obtained in which all orders of scattering anisotropy enter explicitly.. Expressing $q_\ell(z, u)$ in this form makes it possible, in principle, to include anisotropic scattering effects in the neutron slowing down theory to any desired order of approximation.

The optimum age theory involves the approximation of q_o by two terms and q_1 by one term. An extension of the analysis of the slowing down process beyond age type theories was made by utilizing a P_2 approximation to the transport equation and approximating q_o by three terms, q_1 by two terms, and q_2 by one term. With these approximations, it was found that the slowing down process was described by two simultaneous, partial differential equations in the slowing down density and the scattering collision density.

In addition to the input functions $\overline{\mu}_o$ and $\overline{\mu}_o^{\,2}$ which were encountered in the approximate solution of the monoenergetic transport equation, the input functions ξ_{00}, which represents the average logarithmic energy decrement per collision; ξ_{01}, one half the average squared logarithmic energy decrement; and ξ_{10}, the average product of μ_o and the logarithmic energy decrement, occur in the optimum age theory. In order to apply the equations which have been obtained to particular cases, it is necessary to relate the input functions to basic cross section data. Therefore, expressions were derived for the input functions noted above in terms of the experimentally determined angular distribution of the neutron scattering cross section, which has been converted to the center of mass reference frame.

CHAPTER I

INTRODUCTION

The assumption that the neutron scattering is isotropic in the center of mass reference frame is frequently employed in the description of neutron transport phenomena, specifically in reactor theory. However, an examination of the angular distributions of the neutron scattering cross sections given, for example, by Hughes and Carter (1) for materials and neutron energies encountered in reactor calculations indicates that this assumption is not valid in many cases.

A neutron distribution is completely described, except for its state of polarization, by the Boltzmann transport equation. This equation was first obtained by Boltzmann (2) in 1910. An introductory treatment of the transport theory, including the one dimensional spherical harmonics form of the transport equation, is given, e.g., by Glasstone and Edlund (3) and Murray (4). A more complete description of the subject is that of Weinberg and Wigner (5). They give a derivation of the transport equation following the procedure of Boltzmann (2) and obtain the three dimensional spherical harmonics form. Probably the most comprehensive treatment of the transport theory as applied to neutron ensembles is the definitive work of Davison (6).

The transport equation, of course, includes the angular dependence of the scattering cross sections. However, the rather formidable nature of the transport equation precludes the possibility of obtaining exact solutions for realistic situations. Consequently, approximate solutions must be resorted to. The anisotropy of the scattering process is often ignored in arriving at these approximate solutions.

A method of analysis which is often used in nuclear reactor core studies is the Fermi age theory. Elementary treatments of the age theory, based on phenomenological arguments, for isotropic scattering without absorption are given by Soodak and Campbell (7) and the Reactor Handbook (8). The case with weak absorption is described by Murray (9). A more thorough analysis of the effects of absorption is provided by Glasstone and Edlund (10).

The customary age equation represents an approximation to the transport equation in which the neutron scattering is assumed to be isotropic in the center of mass reference frame. The age equation for S wave scattering and no absorption is derived from a P_1 approximation to the transport equation by Marshak, Brooks, and Hurwitz (11). Weinberg and Wigner (12) have improved the procedure of Marshak, et al., to include absorption.

The anisotropy of the neutron scattering process can be introduced into nuclear reactor theory by means of what may be called theoretical input functions. The primary purpose of this research was to develop general relations between some frequently used input functions and the experimentally determined angular distributions, converted to the center of mass system, of the neutron scattering cross sections. Such input functions as the average logarithmic energy decrement, the average cosine, and the average cosine squared of the scattering angle in the laboratory reference frame were considered.

As an application, a solution of the P_2 approximation to the mono-energetic transport equation including second order (in P_ℓ) anisotropic scattering was obtained.

A further goal of this research was to develop an improved age theory which would include the effects of S, P, and D wave scattering. In addition, an extension of the neutron slowing down theory beyond age type theories was considered.

In the course of the analysis of the slowing down process, it was found necessary to examine the effect of the anisotropy of the scattering process on the slowing down functions, q_ℓ, for the one dimensional case. The function q_o is the slowing down density and q_1 represents the slowing down current density. An expression was derived for q_ℓ in terms of the scattering collision density in which all orders of scattering anisotropy entered explicitly.

In addition to the input function ξ_{00}, the average logarithmic energy decrement which has already been mentioned, the input functions ξ_{01} and ξ_{10} were encountered in the development of the improved age theory. These input functions, which can be interpreted physically as one half the average square of the logarithmic energy decrement and the average product of the logarithmic energy decrement with the cosine of the scattering angle in the laboratory system respectively, were also related to the cross section data.

CHAPTER II

NONRELATIVISTIC THEORY OF NEUTRON SCATTERING

In order to investigate the anisotropy of the scattering process in the energy range of interest here, it is first necessary to examine the general nonrelativistic theory of neutron scattering. The treatment which is given will essentially follow the procedure of Glasstone and Edlund (13) up to the point where the transference function is introduced.

Mechanics of the Collision Process.--Diagrams of the scattering process in the laboratory (L) reference frame and the center of mass (C) reference frame are shown in Figure 1. In the L system, the neutron with mass number one moves with a speed V_1 toward a stationary nucleus which has a mass number A. The speed V_m of the center of mass in the L system, i.e., with respect to the stationary nucleus, can be obtained from the momentum balance

$$(A + 1)V_m = V_1 \ . \tag{1}$$

Therefore,

$$V_m = \frac{V_1}{A + 1} \ . \tag{2}$$

Since the center of mass is defined to be at rest in the C system, the nucleus must approach the center of mass with a speed V_m. Then, since the speed of the neutron relative to the nucleus before the collision

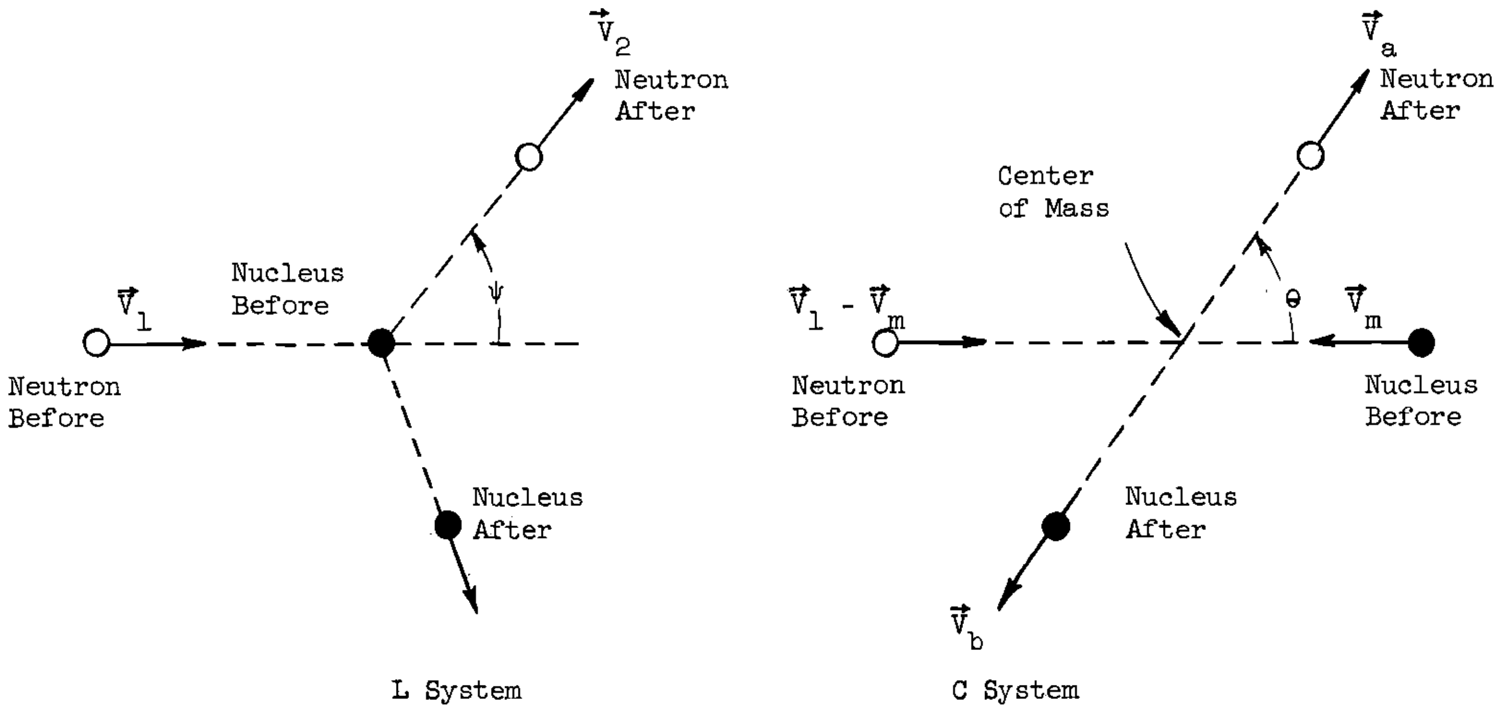


Figure 1. Neutron Scattering in the Laboratory (L) and Center of Mass (C) Systems

is V_1, the speed relative to the center of mass is $V_1 - V_m$. The speed of the neutron before the collision in the C system can be written, by using equation (2), as

$$V_1 - V_m = V_1 - \frac{V_1}{A + 1}$$

$$= \frac{AV_1}{A + 1} \, . \tag{3}$$

In the C system, the momentum of the neutron before the collision is $\frac{AV_1}{A + 1}$ while the momentum of the nucleus is $\frac{AV_1}{A + 1}$ in the opposite direction. After the collision, one has

$$V_a = AV_b \tag{4}$$

where V_a is the final neutron speed and V_b is the final recoil nucleus speed in the C system.

For perfectly elastic collisions, the condition for the conservation of energy in the C system is expressed by

$$\frac{1}{2} (V_1 - V_m)^2 + \frac{1}{2} AV_m{}^2 = \frac{1}{2} V_a{}^2 + \frac{1}{2} AV_b{}^2$$

or

$$\left(\frac{AV_1}{A + 1}\right)^2 + A\left(\frac{V_1}{A + 1}\right)^2 = V_a{}^2 + AV_b{}^2 \, . \tag{5}$$

Solving equations (4) and (5) for V_a and V_b yields

$$V_a = \frac{AV_1}{A + 1} \tag{6}$$

and

$$V_b = \frac{V_1}{A + 1} \, . \tag{7}$$

The speeds of the neutron and the nucleus in the C system are the same after the collision as before the collision.

Transformation from the C System to the L System.--In order to determine the loss of kinetic energy of the neutron resulting from the elastic collision, it is necessary to transform the results obtained in the C system back to the L system. Nonrelativistically, this transformation is accomplished by first observing that the relative velocity between the two systems is $\vec{V}_m$. Therefore, the velocity of the neutron, after collision, in the L system is obtained by adding the velocity $\vec{V}_m$ of the center of mass in the L system to the velocity $\vec{V}_a$ of the neutron after collision in the C system. From Figure 2, by the law of cosines,

$$V_2^{\,2} = V_m^{\,2} + V_a^{\,2} + 2 \, V_m \, V_a \cos \theta \tag{8}$$

where $\vec{V}_2$ is the final neutron velocity in the L system. Introducing V_m and V_a from equations (2) and (6) yields

$$\begin{aligned} V_2^{\,2} &= \left(\frac{V_1}{A + 1}\right)^2 + \left(\frac{AV_1}{A + 1}\right)^2 + \frac{2 \, A \, V_1^{\,2} \cos \theta}{(A + 1)^2} \\ &= \frac{V_1^{\,2}(A^2 + 2 \, A \cos \theta + 1)}{(A + 1)^2} \, . \end{aligned} \tag{9}$$

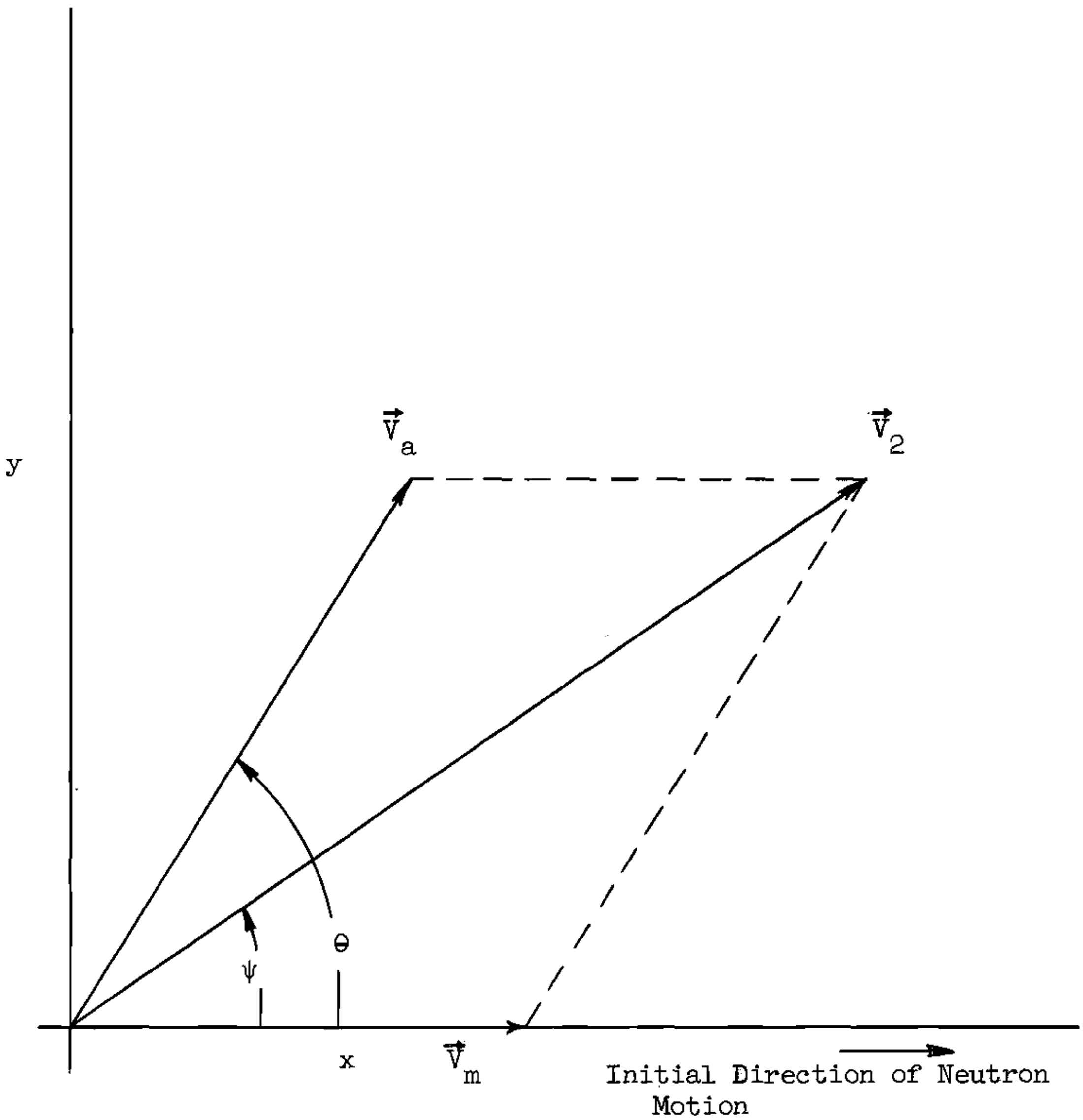


Figure 2. Scattering Angles in the Laboratory and Center of Mass Systems

The relation between the scattering angles θ in the C system and ψ in the L system is obtained by observing that in Figure 2

$$V_2 \cos \psi = V_a \cos \theta + V_m . \tag{10}$$

Then, substituting V_m and V_a from equations (2) and (6) gives

$$V_2 \cos \psi = \frac{AV_1}{A + 1} \cos \theta + \frac{V_1}{A + 1} . \tag{11}$$

Also, from equation (9),

$$V_2 = \frac{V_1}{A + 1} \sqrt{A^2 + 2 A \cos \theta + 1} . \tag{12}$$

Eliminating V_2 between equations (11) and (12) yields

$$\cos \psi = \frac{A \cos \theta + 1}{\sqrt{A^2 + 2 A \cos \theta + 1}} . \tag{13}$$

If, for convenience, the quantities

$$\mu \equiv \cos \theta \tag{14}$$

and

$$\mu_o \equiv \cos \psi \tag{15}$$

are defined, equation (13) can be written as

$$\mu_o = \frac{A\mu + 1}{\sqrt{A^2 + 2 A\mu + 1}} . \tag{16}$$

Energy Change due to Scattering.--The ratio of the kinetic energy, E, of the neutron after scattering to the kinetic energy E' before the collision is

$$\frac{E}{E'} = \frac{{V_2}^2}{{V_1}^2} \; . \tag{17}$$

Using $\frac{V_2}{V_1}$ from equation (12) gives

$$\frac{E}{E'} = \frac{A^2 + 2\ A \cos\ \theta + 1}{(A + 1)^2} \; . \tag{18}$$

If the parameter α defined by

$$\alpha \equiv \left(\frac{A - 1}{A + 1}\right)^2 \tag{19}$$

is introduced, equation (18) becomes

$$\frac{E}{E'} = \frac{1}{2}\left[(1 + \alpha) + (1 - \alpha)\cos\ \theta\right] \; . \tag{20}$$

The maximum value of $\frac{E}{E'}$ occurs for $\theta = 0$ or $\cos\ \theta = 1$ and is given by

$$\left.\frac{E}{E'}\right)_{max} = 1 \; . \tag{21}$$

The minimum value of $\frac{E}{E'}$, which occurs for $\theta = \pi$ or $\cos\ \theta = -1$, is

$$\left.\frac{E}{E'}\right)_{max} = \alpha \; . \tag{22}$$

Differential Scattering Cross Section.--The function $f = f(\mu)$, where $f(\mu)d\mu$ represents the fraction of all scattering collisions for a particular initial energy E' which result in scattering angles in the C system whose cosines lie between μ and $\mu + d\mu$, will be considered. The continuous function f can be expanded in an infinite series of Legendre polynomials in μ of the form

$$f(\mu) = \sum_{n=o}^{\infty} \frac{2n+1}{2} a_n P_n(\mu) \ . \tag{23}$$

The expansion coefficients, a_n, can be evaluated by multiplying both sides of equation (23) by $P_\ell(\mu)d\mu$ and integrating from -1 to 1. This gives

$$\int_{-1}^{1} f(\mu)P_\ell(\mu)d\mu = \sum_{n=o}^{\infty} \frac{2n+1}{2} a_n \int_{-1}^{1} P_\ell(\mu)P_n(\mu)d\mu \ . \tag{24}$$

The orthogonality relation for the Legendre polynomials,

$$\int_{-1}^{1} P_\ell(\mu)P_n(\mu)d\mu = \frac{2}{2\ell+1} \delta_{\ell n} \tag{25}$$

can be introduced to give

$$a_\ell = \int_{-1}^{1} f(\mu)P_\ell(\mu)d\mu$$

$$= \overline{P}_\ell(\mu) \tag{26}$$

where $\overline{P}_\ell(\mu)$ is the mean value of $P_\ell(\mu)$.

The first three expansion coefficients are then

$$a_o = \overline{P}_o(\mu) = 1$$

$$a_1 = \overline{P}_1(\mu) = \overline{\mu}$$

$$a_2 = \overline{P}_2(\mu) = \frac{1}{2}(3\overline{\mu^2} - 1) \ . \tag{27}$$

The expansion (23) becomes

$$f(\mu) = \sum_{\ell=o}^{\infty} \frac{2\ell + 1}{2} \overline{P}_\ell(\mu) \ P_\ell(\mu) \ . \tag{28}$$

This gives, to three terms,

$$f(\mu) = \frac{1}{2} + \frac{3}{2} \overline{\mu}\mu + \frac{5}{2}\left(\frac{3 \overline{\mu^2} - 1}{2} \right) \left(\frac{3 \mu^2 - 1}{2} \right). \tag{29}$$

The differential scattering cross section, $\sigma_s(\mu)$, is related to $f(\mu)$ by

$$\sigma_s(\mu) = \frac{\sigma_s}{2\pi} f(\mu) \tag{30}$$

where σ_s is the total differential scattering cross section given by

$$\sigma_s = \int_{-1}^{1} \sigma_s(\mu) d\mu \ . \tag{31}$$

Consequently, from equation (26),

$$\overline{P}_\ell(\mu) = \frac{2}{\sigma_s} \int_{-1}^{1} \sigma_s(\mu) P_\ell(\mu) d\mu \ . \tag{32}$$

The $\overline{P}_\ell(\mu)$ can be calculated by using for $\sigma_s(\mu)$ experimental data, converted to the center of mass reference frame, for the angular distribution of the neutron scattering cross section in the particular case of interest. A compilation of the angular distributions obtained by various researchers has been made by Hughes and Carter (1). It should be noted that $\sigma_s(\mu)$ is actually a function of the initial neutron energy, E', as well as of μ. Therefore, the $\overline{P}_\ell(\mu)$ are not constants, but are functions of E'.

Macroscopic Scattering Cross Section.--The macroscopic scattering cross section, $\Sigma_s(E')$, can be expressed as

$$\Sigma_s(E') = \int_{-1}^{1} \Sigma_s(E', \mu_o)d\mu_o = \int_{-1}^{1} \Sigma_s(E', \mu)d\mu \ . \tag{33}$$

It is often convenient to express the functions $\Sigma_s(E', \mu_o)$ and $\Sigma_s(E', \mu)$ in terms of infinite series of Legendre polynomials. The function $\Sigma_s(E', \mu_o)$ can be written

$$\Sigma_s(E', \mu_o) = \sum_{\ell=o}^{\infty} \frac{2\ell+1}{4\pi} \Sigma_{s\ell}(E')P_\ell(\mu_o) \ . \tag{34}$$

The expansion coefficients, Σ_s (E'), can be obtained by multiplying $P_m(\mu_o)$ and integrating. This gives

$$\int_{-1}^{1} \Sigma_s(E',\mu_o)P_m(\mu_o)d\mu_o = \sum_{\ell=o}^{\infty} \frac{2\ell+1}{4\pi} \Sigma_{s\ell}(E') \int_{-1}^{1} P_\ell(\mu_o)P_m(\mu_o)d\mu_o \ . \tag{35}$$

Employing the orthogonality relation for the Legendre polynomials leads to

$$\Sigma_{s\ell}(E') = 2\pi \int_{-1}^{1} \Sigma_s(E',\mu_o)P_\ell(\mu_o)d\mu_o$$

$$= \Sigma_s(E')\overline{P}_\ell(E',\mu_o) \ . \tag{36}$$

Equation (33) now becomes

$$\Sigma_s(E',\mu_o) = \sum_{\ell=o}^{\infty} \frac{2\ell+1}{4\pi} \Sigma_s(E')\overline{P}_\ell(E',\mu_o)P_\ell(\mu_o) \ . \tag{37}$$

Similarly, $\Sigma_s(E',\mu)$ can be written as

$$\Sigma_s(E',\mu) = \sum_{\ell=o}^{\infty} \frac{2\ell+1}{4\pi} \Sigma_s(E')\overline{P}_\ell(E',\mu)P_\ell(\mu) \ . \tag{38}$$

The Transference Function.--The expression $\Sigma_s(\vec{r},\ E'\rightarrow E,\ \hat{\Omega}'\rightarrow\hat{\Omega})$, which is the macroscopic scattering cross section at a position $\vec{r}$ for scattering from an initial energy E' and direction $\hat{\Omega}'$ into a unit energy range about E and a unit solid angle about $\hat{\Omega}$, is called the transference function. The vector $\hat{\Omega}'$ is a unit vector in the direction of neutron motion. It should be noted that for isotropy at $\vec{r}$ the functional dependence of the transference function can be expressed by

$$\Sigma_s(\vec{r},\ E'\rightarrow E,\ \hat{\Omega}'\rightarrow\hat{\Omega}) = \Sigma_s(\vec{r},\ E'\rightarrow E,\ \hat{\Omega}'\cdot\hat{\Omega}) = \Sigma_s(\vec{r},\ E'\rightarrow E,\ \mu_o) \ . \tag{39}$$

The transference function can be expanded in an infinite series of Legendre polynomials of the form

$$\Sigma_s(\vec{r},\ E'\to E,\ \hat{\Omega}'\to\hat{\Omega}) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi}\,\Sigma_{s\ell}(\vec{r},\ E'\to E)P_\ell(\mu_o)\ . \tag{40}$$

The quantities E', E, $\hat{\Omega}'$, and $\hat{\Omega}$ cannot all be independent. Conservation of energy and momentum requires that

$$\Sigma_s(\vec{r},\ E'\to E,\ \hat{\Omega}'\to\hat{\Omega}) = \Sigma_s(\vec{r},\ E'\to E)\ \frac{\delta[\mu_o - g(E',\ E)]}{2\pi}\ . \tag{41}$$

The function $g(E',\ E)$ determines the scattering angle in the L system in terms of the initial and final neutron energies. Since the Dirac delta function in equation (41) can be expressed as

$$\delta[\mu_o - g(E',\ E)] = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2}\,P_\ell(\mu_o)P_\ell[g(E',\ E)]\ , \tag{42}$$

the expansion coefficients, $\Sigma_{s\ell}$, in equation (40) are

$$\Sigma_{s\ell}(\vec{r},\ E'\to E) = \Sigma_s(\vec{r},\ E'\to E)P_\ell[g(E',\ E)]\ . \tag{43}$$

Equation (40) then becomes

$$\Sigma_s(\vec{r},\ E'\to E,\ \hat{\Omega}'\to\hat{\Omega}) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi}\,\Sigma_s(\vec{r},\ E'\to E)P_\ell[g(E',\ E)]P_\ell(\mu_o)\ . \tag{44}$$

This equation is valid for inelastic as well as for elastic collisions.

In order to determine the function $g(E',\ E)$ for the elastic scattering case, it is necessary to examine the relation between the scattering angles in the laboratory system and the center of mass system.

Introducing V_m from equation (2) and V_a from equation (6) into equation (10) yields

$$V_2\mu_o = \left[\frac{A\mu + 1}{A + 1} \right] V_1 . \tag{45}$$

Observing that

$$\frac{V_2}{V_1} = \sqrt{\frac{E}{E'}} \tag{46}$$

and solving equation (45) for μ gives

$$\mu = \frac{A + 1}{A} \mu_o \sqrt{\frac{E}{E'}} - \frac{1}{A} . \tag{47}$$

Substituting μ from equation (18) into equation (47) and solving for μ_o yields

$$\mu_o = \frac{(A + 1) \frac{E}{E'} - (A - 1)}{2\sqrt{\frac{E}{E'}}} . \tag{48}$$

The function $g(E', E)$ for elastic scattering is then

$$g(E', E) = \frac{(A + 1) \frac{E}{E'} - (A - 1)}{2\sqrt{\frac{E}{E'}}} . \tag{49}$$

For inelastic scattering, $g(E', E)$ is a more complex function and depends, in general, on the energy of the excited level attained by the scattering nucleus. In this case, the conservation of energy relation for the scattering process is

$$\frac{1}{2}(V_1 - V_m)^2 + \frac{1}{2} AV_m^{\ 2} = \frac{1}{2} V_a^{\ 2} + \frac{1}{2} AV_b^{\ 2} + E_e$$

or

$$\left(\frac{AV_1}{A+1}\right)^2 + A\left(\frac{V_1}{A+1}\right)^2 = V_a^{\ 2} + AV_b^{\ 2} + 2E_e \tag{50}$$

where E_e is the energy of the excited level of the scattering nucleus. From equation (4),

$$V_b = \frac{1}{A} V_a . \tag{51}$$

Eliminating V_b between equations (50) and (51) and solving for V_a yields

$$V_a = \sqrt{\left(\frac{AV_1}{A+1}\right)^2 + \frac{2A}{A+1}\ E_e} \ . \tag{52}$$

If it is noted that

$$\frac{1}{2} V_1^{\ 2} = E \ ,$$

equation (52) can be written as

$$V_a = \frac{AV_1}{A+1}\sqrt{1 + \frac{A+1}{A}\ \frac{E_e}{E}} \ . \tag{53}$$

Substituting V_m from equation (2) and V_a from equation (53) into equation (8) and solving for μ gives

$$\mu = \frac{(A+1)^2}{2A\sqrt{1 + \frac{A+1}{A}\frac{E_e}{E}}}\left[\frac{E}{E'} - \frac{(1+A^2)}{(A+1)^2} - \frac{A}{A+1}\frac{E_e}{E}\right] . \tag{54}$$

Also, in this case, equation (10) becomes

$$V_2\mu_o = \frac{AV_1}{A+1}\,\mu\,\sqrt{1 + \frac{A+1}{A}\frac{E_e}{E}} + \frac{V_1}{A+1} . \tag{55}$$

Introducing μ from equation (54) into equation (55) leads to

$$\mu_o = \frac{(A+1)\frac{E}{E'} - (A-1) - \frac{A}{A+1}\frac{E_e}{E}}{2\sqrt{\frac{E}{E'}}} . \tag{56}$$

Therefore, in the inelastic scattering case, the function $g(E', E)$ has the form

$$g(E', E) = \frac{(A+1)\frac{E}{E'} - (A-1) - \frac{A}{A+1}\frac{E_e}{E}}{2\sqrt{\frac{E}{E'}}} . \tag{57}$$

It appears, from the form of equation (48), that a neutron which is scattered through a given angle, ψ, has two possible final energies. This situation will be examined graphically. From Figure 2 it can be observed that

$$\vec{V}_2 = \vec{V}_m + \vec{V}_a$$

$$= (V_m + V_a \cos\theta)\hat{i} + (V_a \sin\theta)\hat{j}$$

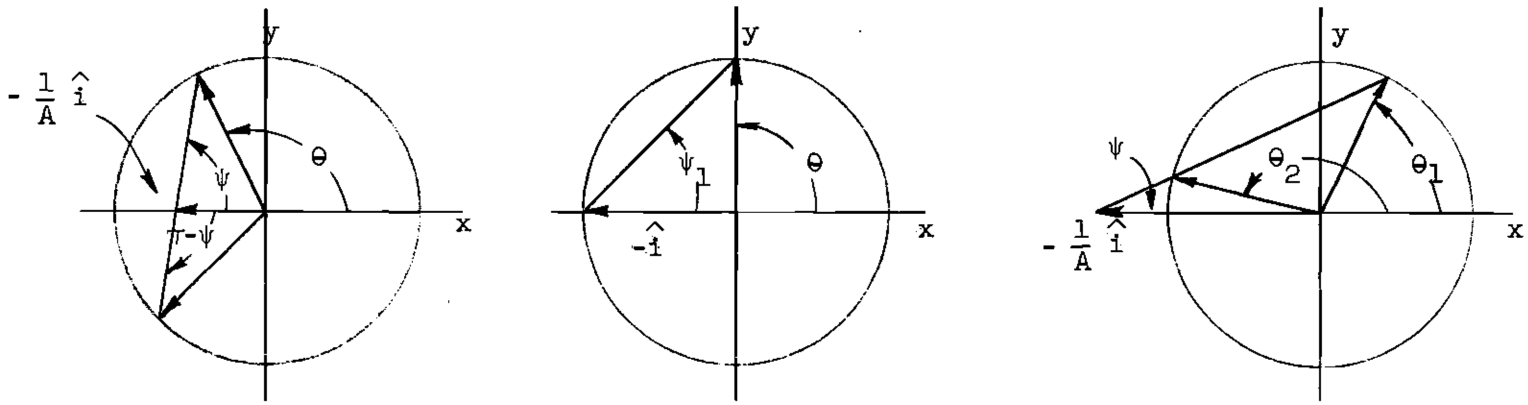
$$= \frac{AV_1}{A+1}\left[(\frac{1}{A} + \cos\theta)\hat{i} + \sin\theta\,\hat{j} \right] \tag{58}$$

where $\hat{i}$ and $\hat{j}$ are unit vectors in the x and y directions respectively. Consequently, a vector in the direction of $\vec{V}_2$ can be obtained by adding a vector $\frac{1}{A}\hat{i}$ to a unit vector in the direction of $\vec{V}_a$.

Now, let the origin of the C system be represented by the center of a unit circle. In this representation, the scattered particle moves along a unit radius vector. The scattering angle ψ is the angle between the x axis and a vector which is the vector sum of a unit radius vector and the vector $\frac{1}{A}\hat{i}$.

In general, three cases must be considered. For the case $A > 1$, Figure 3A applies. It is apparent from an examination of Figure 3A that there can be only one value of θ for each value of ψ as long as $A > 1$. Therefore, one of the two values which would be obtained for E from equation (48) is extraneous in this case.

If $A = 1$, as is essentially the case in the scattering of neutrons by hydrogen nuclei, Figure 3B is appropriate. There are now two acceptable roots of equation (48). One root is always associated with $\theta = 180°$. A scattering angle of 180° in the C system implies that one of the particles involved in the collision continues in the forward direction with the full incident energy while the other remains at rest. The other root is always associated with $\theta = 90°$.



3A 3B 3C

Figure 3. Neutron Scattering Diagrams

Figure 3C shows the case where $A < 1$. There are again two possible final neutron energies for a given angle ψ corresponding to the two possible values of θ. This case has very little practical significance as far as neutron scattering is concerned.

It is frequently desirable to consider, instead of the energy, the variable u defined by

$$u = \ln \frac{E_o}{E} \tag{59}$$

where E_o is a reference energy level. The variable u is called the lethargy. Equation (49) can be written in terms of lethargy as

$$g(u', u) = \frac{A+1}{2} e^{\left(\frac{u'-u}{2}\right)} - \frac{A-1}{2} e^{-\left(\frac{u'-u}{2}\right)} . \tag{60}$$

Also, equations (43) and (44) can be expressed as

$$\Sigma_{s\ell}(u' \to u) = \Sigma_s(u' \to u) P_\ell [g(u', u)] \tag{61}$$

and

$$\Sigma_s(u' \to u, \hat{\Omega}' \to \hat{\Omega}) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \Sigma_s(u' \to u) P_\ell [g(u', u)] P_\ell(\mu_o) \tag{62}$$

where, for simplicity, the positional dependence has been suppressed.

The description of the transference function can be completed by investigating the function $\Sigma_s(u' \to u)$. This function can be written as

$$\Sigma_s(u' \to u) = - 2\pi \, \Sigma_s(u', u) \frac{d\mu}{du} . \tag{63}$$

Eliminating μ_o between equations (47) and (48) leads to

$$\mu = \frac{(A+1)^2}{2A}\,\frac{E}{E'} - \frac{(A^2+1)}{2A}$$

$$= \frac{2\,e^{u'-u} - (1+\alpha)}{1-\alpha}\ . \tag{64}$$

Therefore,

$$\frac{d\mu}{du} = -\,\frac{2\,e^{u'-u}}{1-\alpha}\ . \tag{65}$$

From equation (38),

$$\Sigma_s(u',\ \mu) = \sum_{\ell=o}^{\infty} \frac{2\ell+1}{4\pi}\,\Sigma_s(u')\overline{P}_\ell(u',\ \mu)P_\ell(\mu)\ . \tag{66}$$

Introducing the expressions (65) and (66) into equation (63) yields

$$\Sigma_s(u'\to u) = \frac{e^{u'-u}}{1-\alpha}\,\Sigma_s(u')\sum_{\ell=o}^{\infty}(2\ell+1)\overline{P}_\ell(u',\ \mu)P_\ell(\mu)\ . \tag{67}$$

Equations (61) and (62) now become

$$\Sigma_{s\ell}(u'\to u) = \frac{e^{u'-u}}{1-\alpha}\,\Sigma_s(u')P_\ell[g(u',\ u)]\sum_{n=o}^{\infty}(2n+1)\overline{P}_n(u',\ \mu)P_n(\mu) \tag{68}$$

and

$$\Sigma_s(u'\to u,\ \hat{\Omega}'\to\hat{\Omega}) = \frac{e^{u'-u}}{1-\alpha}\,\Sigma_s(u')\sum_{\ell,n=o}^{\infty}\frac{(2\ell+1)(2n+1)}{4\pi}\,P_\ell[g(u',u)]P_\ell(\mu_o)\overline{P}_n(u',\mu)P_n(\mu). \tag{69}$$

CHAPTER III

THE BOLTZMANN TRANSPORT EQUATION

Apart from its state of polarization, a neutron ensemble is completely described at any given time, t, by its distribution in six dimensional phase space (x, y, z, V_x, V_y, V_z). Considerations based on neutron distributions in ordinary space do not represent a realistic treatment.

For neutrons, it is convenient to select as the six coordinates required to specify a point in phase space the position vector $\vec{r}$; the kinetic energy of the neutron, E; and a unit vector $\hat{\Omega}$ in the direction of motion of the neutron. The vector $\hat{\Omega}$ is usually specified by the polar angle δ and the azimuth angle ϕ.

It is customary to consider the angular flux $F(\vec{r}, E, \hat{\Omega}, t)$ as the fundamental variable of the theory. The angular flux is the number of neutrons per unit volume about $\vec{r}$ and in a unit energy interval about E with directions of motion in a unit solid angle about $\hat{\Omega}$ multiplied by the neutron speed V corresponding to the energy E.

The time dependent distribution function $N(\vec{r}, E, \hat{\Omega}, t)$, which represents the number of neutrons in a unit volume of phase space, is obviously related to the angular flux by

$$N = \frac{F}{V} \; . \tag{70}$$

Derivation from a Statistical Point of View.--The Boltzmann transport equation can be derived by considering the flow of neutrons in and out of a differential volume $d\vec{r}\, dE\, d\hat{\Omega}$ of phase space. The viewpoint and derivation which will be presented is based on that of Weinberg and Wigner (14).

The coordinates of the neutrons in phase space are changed by two mechanisms. The ordinary space coordinates (x, y, z) change due to the uniform motion of the neutrons. This effect does not influence E or $\hat{\Omega}$. Secondly, E and $\hat{\Omega}$ are altered as a result of neutron collisions while x, y, and z remain unchanged.

The flow of neutrons in phase space associated with the first effect produces a current $\hat{\Omega}F(\vec{r}, E, \hat{\Omega}, t)$. In the differential element $d\vec{r}\, dE\, d\hat{\Omega}$, the change in the distribution function resulting from this current is

$$\left.\frac{\partial N}{\partial t}\right)_1 = - \vec{\nabla} \cdot (\hat{\Omega}F)$$

$$= - \hat{\Omega}_x \frac{\partial F}{\partial x} - \hat{\Omega}_y \frac{\partial F}{\partial y} - \hat{\Omega}_z \frac{\partial F}{\partial z}$$

$$= - \hat{\Omega} \cdot \vec{\nabla} F \tag{71}$$

where the operator $\vec{\nabla}$ operates only on the ordinary space coordinates.

The change in N produced by collision processes which affect the number of neutrons in $d\vec{r}\, dE\, d\hat{\Omega}$ is

$$\left.\frac{\partial N}{\partial t}\right)_2 = \int dE' d\hat{\Omega}' F(\vec{r}, E', \hat{\Omega}', t)\ \Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \to \Omega)$$

$$- \int dE' d\hat{\Omega}' F(\vec{r}, E, \hat{\Omega}, t)\ \Sigma_s(\vec{r}, E \to E', \hat{\Omega} \to \hat{\Omega}')$$

$$- F(\vec{r}, E, \hat{\Omega}, t)\ \Sigma_a(\vec{r}, E). \tag{72}$$

The first term on the right side of equation (72) represents scattering into the volume element considered; the second term, scattering out; and the last term, absorption. The function $\Sigma_s(\vec{r}, E'\to E, \hat{\Omega}'\to\hat{\Omega})$ is the transference function which has been discussed previously. Since

$$\int dE' d\hat{\Omega}' \Sigma_s(\vec{r}, E\to E', \hat{\Omega}\to\hat{\Omega}') = \Sigma_s(\vec{r}, E, \hat{\Omega}), \tag{73}$$

equation (72) becomes

$$\left.\frac{\partial N}{\partial t}\right)_2 = \int dE' d\hat{\Omega}' F(\vec{r}, E', \hat{\Omega}', t)\, \Sigma_s(\vec{r}, E'\to E, \hat{\Omega}'\to\hat{\Omega})$$

$$- F(\vec{r}, E, \hat{\Omega}, t)\Sigma(\vec{r}, E, \hat{\Omega}) \tag{74}$$

where

$$\Sigma(\vec{r}, E, \hat{\Omega}) = \Sigma_s(\vec{r}, E, \hat{\Omega}) + \Sigma_a(\vec{r}, E) \tag{75}$$

is the total cross section.

A complete neutron balance for the differential volume $d\vec{r}\, dE\, d\hat{\Omega}$, including a true source term, S, is

$$\frac{\partial N}{\partial t}(\vec{r}, E, \hat{\Omega}, t) = - \hat{\Omega}\cdot\vec{\nabla}F(\vec{r}, E, \hat{\Omega}, t) + S(\vec{r}, E, \hat{\Omega})$$

$$- F(\vec{r}, E, \hat{\Omega}, t)\Sigma(\vec{r}, E, \hat{\Omega})$$

$$+ \int dE' d\hat{\Omega}' F(\vec{r}, E', \hat{\Omega}', t)\Sigma_s(\vec{r}, E'\to E, \hat{\Omega}'\to\hat{\Omega}) \; . \tag{76}$$

If only the stationary distribution $N(\vec{r}, E, \hat{\Omega})$ is considered and it is assumed that the total cross section Σ is independent of $\hat{\Omega}$, equation (76) becomes

$$\hat{\Omega}\cdot\nabla F(\vec{r},\ E,\ \hat{\Omega}) + F(\vec{r},\ E,\ \hat{\Omega})\ \Sigma(\vec{r},\ E)$$
$$-\int dE'd\hat{\Omega}'F(\vec{r},\ E',\ \hat{\Omega}')\ \Sigma_s(\vec{r},\ E'\to E,\ \hat{\Omega}'\to\hat{\Omega}) = S(\vec{r},\ E,\ \hat{\Omega})\ . \qquad (77)$$

Equation (77) is one form of the Boltzmann transport equation. However, it is frequently advantageous to change the form of the transport equation.

Spherical Harmonics Form.--A particularly useful form of the Boltzmann transport equation can be obtained by applying the spherical harmonics method to equation (77). The transference function can be represented by the infinite series of Legendre polynomials in μ_o given by equation (40) which reads

$$\Sigma_s(r,\ E'\to E,\ \hat{\Omega}'\to\hat{\Omega}) = \sum_{\ell=o}^{\infty} \frac{2\ell+1}{4\pi}\ \Sigma_{s\ell}(\vec{r},\ E'\to E)P_\ell(\mu_o)\ .$$

The expansion coefficients $\Sigma_{s\ell}(\vec{r},\ E'\to E)$ are given by equation (68). The series expansions

$$F(\vec{r},\ E,\ \hat{\Omega}) = \sum_{\ell=o}^{\infty}\ \sum_{m=-\ell}^{\infty} \frac{2\ell+1}{4\pi}\ F_{\ell m}(\vec{r},\ E)\ Y_{\ell m}(\hat{\Omega}) \qquad (78)$$

and

$$S(\vec{r},\ E,\ \hat{\Omega}) = \sum_{\ell=o}^{\infty}\ \sum_{m=-\ell}^{\infty} \frac{2\ell+1}{4\pi}\ S_{\ell m}(\vec{r},\ E)\ Y_{\ell m}(\hat{\Omega}) \qquad (79)$$

will be employed for the angular flux function F and the source term S. The functions $Y_{\ell m}(\hat{\Omega})$ are the associate spherical harmonics defined by

$$Y_{\ell m}(\hat{\Omega}) = \frac{e^{im\phi}(\sin\delta)^{-m}}{2^{\ell}\,\ell!} \quad \frac{(\ell+m)!}{(\ell-m)!} \quad \frac{d^{\ell-m}(\cos^2\delta-1)^{\ell}}{(d\cos\delta)^{\ell-m}} \tag{80}$$

where δ is the angle between $\hat{\Omega}$ and the z axis. They satisfy the orthogonality relation

$$\int Y_{\ell' m'}(\hat{\Omega})\, Y^{*}_{\ell m}(\hat{\Omega})\, d\hat{\Omega} = \frac{4\pi}{2\ell+1}\,\delta_{\ell\ell'}\,\delta_{mm'} \tag{81}$$

and the addition theorem

$$P_{\ell}(\mu_o) = \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\Omega})\, Y^{*}_{\ell m}(\hat{\Omega}') \tag{82}$$

where the asterisk indicates complex conjugation.

Substituting the series expansions (40), (78), and (79) into equation (77), multiplying the resulting equation by $Y^{*}_{\ell m}(\hat{\Omega})$, and integrating leads to

$$\sqrt{\frac{(\ell+2+m)(\ell+1+m)}{2\ell+1}}\left(-\frac{1}{2}\frac{\partial}{\partial x}F_{\ell+1,m+1} - \frac{i}{2}\frac{\partial}{\partial y}F_{\ell+1,m+1}\right)$$

$$+\sqrt{\frac{(\ell+1-m)(\ell+2-m)}{2\ell+1}}\left(\frac{1}{2}\frac{\partial}{\partial x}F_{\ell+1,m-1} - \frac{i}{2}\frac{\partial}{\partial y}F_{\ell+1,m-1}\right)$$

$$+\sqrt{\frac{(\ell-m-1)(\ell-m)}{2\ell+1}}\left(\frac{1}{2}\frac{\partial}{\partial x}F_{\ell-1,m+1} + \frac{i}{2}\frac{\partial}{\partial y}F_{\ell-1,m+1}\right)$$

$$+\sqrt{\frac{(\ell+m)(\ell+m-1)}{2\ell+1}}\left(-\frac{1}{2}\frac{\partial}{\partial x}F_{\ell-1,m-1} + \frac{i}{2}\frac{\partial}{\partial y}F_{\ell-1,m-1}\right)$$

$$+ \sqrt{\frac{(\ell+m+1)(\ell-m+1)}{2\ell+1}} \quad \frac{\partial}{\partial z} F_{\ell+1,m}$$

$$+ \sqrt{\frac{(\ell+m)(\ell-m)}{2\ell+1}} \quad \frac{\partial}{\partial z} F_{\ell-1,m} - \int_0^\infty F_{\ell m}(\vec{r}, E') \, \Sigma_{s\ell}(\vec{r}, E' \to E) dE'$$

$$+ \Sigma(\vec{r}, E) \, F_{\ell m}(\vec{r}, E) = S_{\ell m}(\vec{r}, E) \tag{83}$$

where $i = \sqrt{-1}$. Equation (83) may be called the spherical harmonics form of the Boltzmann transport equation.

The convergence of the series expansion (78) where the $F_{\ell m}$ are the solutions of equation (83) has been established by Davison (15).

The expansion coefficient $F_{oo}(\vec{r}, E)$ is the scalar flux $\Phi(\vec{r}, E)$. The ℓ=1 terms in the expansion of F can be interpreted physically by examining the components of the neutron current density. These components are

$$J_z(\vec{r}, E) = F_{10}(\vec{r}, E) \tag{84}$$

$$J_x(\vec{r}, E) = \frac{1}{\sqrt{2}} \left[F_{1,-1}(\vec{r}, E) - F_{11}(\vec{r}, E) \right] \tag{85}$$

and

$$J_y(\vec{r}, E) = -\frac{1}{\sqrt{2}} \left[F_{1,-1}(\vec{r}, E) + F_{11}(\vec{r}, E) \right] . \tag{86}$$

One Dimensional Case.--Considerable simplification occurs in equation (83) if $S_{\ell m}$ and the cross sections are functions of only one space coordinate.

If $S_{\ell m} = S_{\ell m}(z, E)$, $\Sigma_{s\ell} = \Sigma_{s\ell}(z, E' \to E)$, and $\Sigma = \Sigma(z, E)$, equation (83) reduces to the one dimensional transport equation

$$\frac{(\ell+m+1)(\ell-m+1)}{2\ell+1} \frac{\partial}{\partial z} F_{\ell+1,m}(z,E) + \frac{(\ell+m)(\ell-m)}{2\ell+1} \frac{\partial}{\partial z} F_{\ell-1,m}(z,E)$$

$$- \int_0^\infty F_{\ell m}(z, E')\Sigma_{s\ell}(z, E' \to E)dE + \Sigma(z, E)F_{\ell m}(z, E)$$

$$= S_{\ell m}(z, E) \quad . \tag{87}$$

A further simplification results if there is no dependence on the azimuth ϕ. In this case, only one value of m, m=o, occurs. Since the double subscript notation is now superfluous, equation (87) can be written as

$$\frac{\ell+1}{2\ell+1} \frac{\partial}{\partial z} F_{\ell+1}(z, E) + \frac{\ell}{2\ell+1} \frac{\partial}{\partial z} F_{\ell-1}(z, E)$$

$$- \int_0^\infty F_\ell(z, E') \Sigma_{s\ell}(z, E \to E)dE'$$

$$+ \Sigma(z, E) F_\ell(z, E) = S_\ell(z, E) \quad . \tag{88}$$

The functions F_o and F_1 in equation (88) represent the scalar flux $\phi(z, E)$ and the neutron current density in the z direction, $\vec{J}(z, E)$, respectively. It should be noted that the total scalar flux and the total neutron current density are given by

$$\phi(z) = \int \phi(z, E)dE \tag{89}$$

and

$$\vec{J}(z) = \int \vec{J}(z, E)dE \ . \tag{90}$$

Transformation from a Plane Source to a Point Source.--It might be supposed that the applicability of solutions of the one dimensional transport equation would be limited to a few special cases. However, this is not always true. In some instances, it is possible to transform the results of a one dimensional analysis to the three dimensional case.

As an example, the flux due to an isotropic, monoenergetic point source, $\delta(r)$, will be considered. The relation between the flux $\phi_{(\delta r)}(\mathbf{r})$ due to the point source and the flux $\phi_{(\delta z)}(z)$ resulting from an infinite plane source $\delta(z)$ is given by Murray (16) as

$$\phi_{(\delta r)}(r) = - \left[\frac{1}{2\pi z} \frac{d}{dz} \phi_{(\delta z)}(z) \right]_{z=r} \ . \tag{91}$$

The flux produced by any source which can be represented by a superposition of isotropic point sources can be built up from $\phi_{(\delta r)}$. Hence, a solution to the three dimensional transport equation for such a source can be obtained by applying the transformation relation (91) to the solution of the one dimensional transport equation involving a delta function plane source and then superposing suitably the various point source fluxes. There is, then, frequently no loss of generality involved when only the one dimensional case is considered.

CHAPTER IV

AN APPROXIMATE SOLUTION OF THE MONOENERGETIC TRANSPORT EQUATION

A monoenergetic transport equation is often used to describe the behavior of an ensemble of thermalized neutrons. This description is not exact; but since the energy change per collision is small, it represents a good approximation.

Equation (88), with the additional assumptions that all the neutrons have the same energy and that the cross sections are independent of position, is the simplest form of the general transport equation (83). However, even in its simplest form, the transport equation is not usually amenable to exact solutions and approximate solutions must be utilized. An approximate method which is frequently employed is based on the assumption that $F_m = 0$ for $m > \ell$. The transport equation which is consistent with this assumption is called the P_ℓ approximation to the general transport equation.

A P_2 Approximation.--A neutron distribution for which there is no dependence on the azimuth ϕ or on the energy, E, will be considered. It it is further assumed that the cross sections are independent of position, equation (88) reduces to the one dimensional one velocity transport equation

$$\frac{\ell}{2\ell+1} \frac{d}{dz} F_{\ell-1} + (\Sigma - \Sigma_{s\ell})F_\ell + \frac{\ell+1}{2\ell+1} \frac{d}{dz} F_{\ell+1} = S_\ell \; . \tag{92}$$

The $\Sigma_{s\ell}$ in this case are given by equation (36) as

$$\Sigma_{s\ell}(E) = \Sigma_s(E)\overline{P}_\ell(E, \mu_o) \; . \tag{93}$$

Equation (92) actually represents a system of coupled linear ordinary differential equations.

For many applications, the source term S can be considered to be isotropic; i.e., $S_\ell = 0$ for $\ell = 1, 2, 3, \ldots$. The following considerations will be restricted to this case.

Solutions of the set of equations (92) can be readily obtained if the approximation $F_\ell = 0$ for $\ell = 3, 4, 5, \ldots$ is employed. Under these conditions, the set of equations (92) is limited to

$$\Sigma_a F_o + \frac{dF_1}{dz} = S \tag{94}$$

$$\frac{1}{3}\frac{dF_o}{dz} + \Sigma_b F_1 + \frac{2}{3}\frac{dF_2}{dz} = 0 \tag{95}$$

$$\frac{2}{5}\frac{dF_1}{dz} + \Sigma_c F_2 = 0 \tag{96}$$

where

$$S = S_o \tag{97}$$

$$\Sigma_a = \Sigma - \Sigma_{so} \tag{98}$$

$$\Sigma_b = \Sigma - \Sigma_{s1} \tag{99}$$

$$\Sigma_c = \Sigma - \Sigma_{s2} \; . \tag{100}$$

The set of equations (94), (95), and (96) constitute a P_2 approximation to the transport equation (92).

Solution by the Use of the Green's Function.--To facilitate the solution of equations (94), (95), and (96) for the scalar flux $F_o(z) \equiv \Phi(z)$, it is desirable to change the form of these equations. Differentiating equation (94) twice with respect to z gives

$$\Sigma_a \frac{d^2F_o}{dz^2} + \frac{d^3F_1}{dz^3} = \frac{d^2S}{dz^2} \ . \tag{101}$$

Differentiating equation (96) with respect to z leads to

$$\frac{dF_2}{dz} = - \frac{2}{5\Sigma_c} \frac{d^2F_1}{dz^2} \ . \tag{102}$$

Substituting $\frac{dF_2}{dz}$ from equation (102) into equation (95) yields

$$\frac{1}{3} \frac{dF_o}{dz} + \Sigma_b F_1 - \frac{4}{15\Sigma_c} \frac{d^2F_1}{dz^2} = 0 \ . \tag{103}$$

Differentiating equation (103) with respect to z leads to

$$\frac{d^3F_1}{dz^3} = \frac{15\Sigma_c}{4} \left[\frac{1}{3} \frac{d^2F_o}{dz^2} + \Sigma_b \frac{dF_1}{dz} \right] \ . \tag{104}$$

From equation (94),

$$\frac{dF_1}{dz} = S - \Sigma_a F_o \ . \tag{105}$$

Therefore, equation (104) becomes

$$\frac{d^3F_1}{dz^3} = \frac{5\Sigma_c}{4} \frac{d^2F_o}{dz^2} + \frac{15\ \Sigma_b\Sigma_c}{4} (S - \Sigma_a F_o) \ . \tag{106}$$

Substituting $\dfrac{d^3F_1}{dz^3}$ from equation (106) into equation (101) gives

$$-\frac{1}{3\Sigma_b}\left(1 + \frac{4}{5}\frac{\Sigma_a}{\Sigma_c}\right)\frac{d^2F_o}{dz^2} + \Sigma_a F_o = S - \frac{4}{15\ \Sigma_b\Sigma_c} \frac{d^2S}{dz^2} \ . \tag{107}$$

Differentiating equation (105) with respect to z yields

$$\frac{d^2F_1}{dz^2} = \frac{dS}{dz} - \Sigma_a \frac{dF_o}{dz} \ . \tag{108}$$

Substituting $\dfrac{d^2F_1}{dz^2}$ from equation (108) into equation (103) leads to

$$F_1 = -\frac{1}{3\Sigma_b}\left(1 + \frac{4}{5}\frac{\Sigma_a}{\Sigma_c}\right)\frac{dF_o}{dz} + \frac{4}{15\ \Sigma_b\Sigma_c} \frac{dS}{dz} \ . \tag{109}$$

Introducing $\dfrac{dF_1}{dz}$ from equation (105) into equation (96) and solving for F_2 yields

$$F_2 = \frac{2}{5}\frac{\Sigma_a}{\Sigma_c} F_o - \frac{2}{5} \frac{S}{\Sigma_c} \ . \tag{110}$$

The set of equations (94), (95), and (96) will be replaced by the equivalent set (107), (109), and (110). These equations can be written as

$$- D_2 \frac{d^2F_o}{dz^2} + \Sigma_a F_o = S - C \frac{d^2S}{dz^2} \tag{111}$$

$$F_1 = - D_2 \frac{dF_o}{dz} + C \frac{dS}{dz} \tag{112}$$

$$F_2 = \frac{2}{5\Sigma_c} (\Sigma_a F_o - S) \tag{113}$$

where

$$D_2 \equiv \frac{1}{3\Sigma_b}\left(1 + \frac{4}{5} \frac{\Sigma_a}{\Sigma_c} \right) \tag{114}$$

is the diffusion coefficient for the P_2 approximation and

$$C \equiv \frac{4}{15 \Sigma_b\Sigma_c} . \tag{115}$$

Attention will be confined to equation (111). Since D_2 is independent of position, equation (111) can be written in the standard form

$$\frac{d}{dz} (p \frac{dF_o}{dz}) - r F_o = - f(z) \tag{116}$$

of the inhomogeneous Sturm-Liouville equation. The Green's function associated with equation (111) is

$$G(z, z') = \frac{1}{D_2} \int_o^\infty \frac{\cos (z - z')K}{\kappa^2 + K^2} dK$$

$$= \frac{1}{2\kappa D_2} e^{-\kappa |z-z'|} \tag{117}$$

where

$$\kappa^2 \equiv \frac{\Sigma_a}{D_2} . \tag{118}$$

Consequently, the scalar flux $\Phi(z) \equiv F_o(z)$ is given by

$$\Phi(z) = \frac{1}{2\kappa D_2} \int_{-\infty}^{\infty} e^{-\kappa|z-z'|} \left[S(z') - C \frac{d^2 S}{dz'^2} (z') \right] dz'. \tag{119}$$

Integration by parts leads to

$$\Phi(z) = \frac{1-C\kappa^2}{2\kappa D_2} \int_{-\infty}^{\infty} S(z') e^{-\kappa|z-z'|} dz' + \frac{D}{D_2} S(z) . \tag{120}$$

As might be expected from the linearity of equation (111), this solution is in agreement with the principle of linear superposition of sources. The scalar flux can be written as

$$\Phi(z) = \int_{-\infty}^{\infty} \phi_{(\delta z)}(z-z') \, S(z') dz' \tag{121}$$

where

$$\Phi_{(\delta z)}(z) = \frac{1-C\kappa^2}{2\kappa D_2} e^{-\kappa|z|} + \frac{C}{D_2} \delta(z) \tag{122}$$

is the flux due to an isotropic infinite plane source located at $z = 0$. The function $\Phi_{(\delta z)}(z)$ is obtained from equation (119) when $S(z)$ is a Dirac delta function representing an infinite plane source of unit strength. It should not be confused with the Green's function. Also, in performing the integration in equation (120), it should be noted that

$$\lim_{\epsilon \to 0} \left[\frac{dG}{dz} \bigg|_{z=z'+\epsilon} + \frac{dG}{d(-z)} \bigg|_{z=z'-\epsilon} \right] = \frac{1}{p(z')} \quad . \tag{123}$$

In this case, $p = - D_2$.

The procedure leading to equations (121) and (122) has consistently accounted for the source normalization constants.

Up to this point, the analysis has been restricted to the one dimensional case. However, the flux due to an infinite isotropic plane source can be transformed to the flux due to an isotropic point source. Applying the transformation relation (91) to the flux equation (122) yields

$$\Phi_{(\delta r)}(r) = \frac{1-C\kappa^2}{4\pi D_2} \frac{e^{-\kappa r}}{r} + \frac{C}{2\pi D_2} \frac{\delta(r)}{r^2} \, . \tag{124}$$

The flux produced by any source can be built up from $\Phi_{(\delta r)}(r)$.

It should be noted that equations (122) and (124) can be of considerably different behavior than the equivalent solutions in the P_1 approximation. The P_1 approximation is equivalent to the ordinary diffusion theory. Equations (122) and (124) thus represent improved diffusion theory solutions. The ordinary diffusion theory solution corresponding to equation (122) is given by Glasstone and Edlund (17) as

$$\Phi_{(\delta z)}(z) = \frac{e^{-\kappa|z|}}{2\kappa D_1} \tag{125}$$

where

$$D_1 = \frac{1}{3\Sigma[1 - \bar{P}_1(E, \mu_o)]}$$

is the diffusion coefficient in the P_1 approximation for the case where $\Sigma_a \ll \Sigma$.

If the origin is excluded, a comparison of equations (122) and (125) shows that the principal difference is that the improved diffusion theory yields a factor $1 - C\kappa^2$ rather than unity, as given by the ordinary diffusion theory. This difference should result in a more accurate account of the asymptotic behavior of the diffused flux.

For the case of an isotropic infinite plane source located at $z = 0$, it is possible to obtain a rigorous solution of the transport equation. The asymptotic form of this rigorous solution is given by Glasstone and Edlund (18) as

$$\Phi_{(\delta z)}(z) = \frac{\alpha}{2\kappa D} e^{-\kappa|z|} \tag{126}$$

where

$$\alpha \equiv 2 \frac{\Sigma_a}{\Sigma_s} \cdot \frac{\Sigma^2 - \kappa^2}{\kappa^2 - \Sigma \Sigma_a} .$$

If $\Sigma_a \ll \Sigma$, Glasstone and Edlund (19) have shown that

$$\alpha \simeq 1 - \frac{4}{5} \frac{\Sigma_a}{\Sigma} . \tag{127}$$

An examination of equations (122) and (126) for the case where the origin is excluded shows that a comparison between the improved diffusion theory solution and the asymptotic rigorous solution is essentially equivalent to a comparison of the factors $1 - C\kappa^2$ and α. By using the definition of C and assuming that $\Sigma \gg \Sigma_{s2}$, the factor $1 - C\kappa^2$ can be expressed as

$$1 - C\kappa^2 \simeq 1 - \frac{4\kappa^2}{15\Sigma[\Sigma - \Sigma_s\overline{P}_1(E, \mu_o)]}$$

$$\simeq 1 - \frac{4\Sigma_a}{15\Sigma[\Sigma - \Sigma_s\overline{P}_1(E, \mu_o)]\, D_2} \quad . \qquad (128)$$

Introducing D_2 from equation (114) into equation (128) and neglecting the $\frac{\Sigma_a}{\Sigma}$ term associated with D_2 leads to

$$1 - C\kappa^2 \simeq 1 - \frac{4}{5}\ \frac{\Sigma_a}{\Sigma}\ \frac{\Sigma[1 - \overline{P}_1(E, \mu_o)]}{\Sigma - \Sigma_s\overline{P}_1(E, \mu_o)} \quad . \qquad (129)$$

A comparison of the approximate relations (127) and (129) shows that the factors α and $1 - C\kappa^2$ are very nearly equal for the case $\Sigma_a \ll \Sigma$. Consequently, the improved diffusion theory solutions (122) and (124) represent very good approximations to the actual flux in the asymptotic region for a weakly absorbing medium.

Diffusion Coefficient for the P_2 Approximation.--Some approximate expressions for the diffusion coefficient D_2 will now be considered. The defining relation (114) can be written as

$$D_2 \equiv \frac{1}{3(\Sigma - \Sigma_{s1})}\left[\ 1 + \frac{4}{5}\,\frac{\Sigma_a}{\Sigma - \Sigma_{s2}}\ \right] . \qquad (130)$$

For $\Sigma \gg \Sigma_{s2}$ and $\Sigma_a \ll \Sigma$, D_2 can be approximated by

$$D_2 \quad \frac{1}{3(\Sigma - \Sigma_{s1})(1 - \frac{4}{5}\frac{\Sigma_a}{\Sigma})} . \tag{131}$$

Introducing the relation

$$\Sigma_{s1}(E) = \Sigma_s(E)\overline{P}_1(E, \mu_o) \tag{132}$$

from equation (36) into equation (131) yields

$$D_2 \simeq \frac{1}{3\Sigma(1 - \overline{P}_1)(1 - \frac{4}{5}\frac{\Sigma_a}{\Sigma}) + \Sigma_a\overline{P}_1(1 - \frac{4}{5}\frac{\Sigma_a}{\Sigma})} \tag{133}$$

$$\simeq \frac{1}{3\Sigma(1 - \overline{P}_1)(1 - \frac{4}{5}\frac{\Sigma_a}{\Sigma} + \frac{\Sigma_a}{\Sigma}\frac{\overline{P}_1}{1-\overline{P}_1}) - \frac{4}{5}\frac{\Sigma_a{}^2}{\Sigma}\overline{P}_1} \tag{134}$$

$$\simeq \frac{1}{3\Sigma(1 - \overline{P}_1)\left[1 - \frac{\Sigma_a}{\Sigma}(\frac{4}{5} - \frac{P_1}{1-\overline{P}_1})\right]} . \tag{135}$$

with

$$\overline{P}_1 = \overline{P}_1(E, \mu_o) .$$

It is apparent from the form of the approximation (135) that the diffusion coefficient D_2 is equal to the usual P_1 diffusion coefficient if Σ_a is neglected.

CHAPTER V

AGE APPROXIMATION TO TRANSPORT THEORY

The analysis of the slowing down process for fast neutrons in a moderating medium is, of course, a considerably more complicated problem than the study of the behavior of monoenergetic neutrons. This is apparent from the form of the transference function, $\Sigma_s(\vec{r}, E'\to E, \hat{\Omega}'\to\hat{\Omega})$, when the energy dependence must be considered in addition to the angular dependence.

The numerical "multigroup methods" in which the neutrons are divided into energy groups are frequently employed. However, the Fermi "age" theory is often used to obtain analytic expressions which describe the slowing down process. The fundamental postulate of age theory is the assumption that the slowing down of a neutron is a continuous process. This assumption makes it possible to obtain an approximate solution of the energy dependent transport equation. Since the average amount of energy lost by a neutron per collision increases as the mass number, A, of the scattering nucleus decreases, the accuracy of this approximation decreases as A decreases. The fundamental dependent variable of age theory is the slowing down density, q.

Age Theory for S Wave Scattering.--In order to obtain a source term for the equation governing the diffusion of thermalized neutrons, the one dimensional transport equation (88) can often be used. However, the energy dependence of the terms in the transport equation must be considered. Since it is more convenient to use lethargy as a variable rather than energy, a change of variable from E to u will be made in equation (88). The one dimensional lethargy dependent transport equation for the case in which the cross sections are independent of position is then

$$\frac{\ell}{2\ell+1}\frac{\partial}{\partial z}F_{\ell-1}(z, u) + \Sigma F_{\ell}(z, u) - \int_{u-\epsilon}^{u} du'\Sigma_{s\ell}(u'\to u)F_{\ell}(z, u')$$

$$+ \frac{\ell+1}{2\ell+1}\frac{\partial}{\partial z}F_{\ell+1}(z, u) = S_{\ell}(z, u). \qquad (136)$$

It has again been assumed that there is no dependence on the azimuth ϕ. Equation (136) can be written in the form

$$\frac{\ell}{2\ell+1}\frac{\partial}{\partial z}F_{\ell-1}(z, u) + [\Sigma - \Sigma_{s\ell}(u)]F_{\ell}(z, u)$$

$$+ \frac{\partial}{\partial u}\int_{u-\epsilon}^{u} du' \int_{u}^{u'+\epsilon} du''\Sigma_{s\ell}(u'\to u'')F_{\ell}(z, u')$$

$$+ \frac{\ell+1}{2\ell+1}\frac{\partial}{\partial z}F_{\ell+1}(z, u) = S_{\ell}(z, u) \qquad (137)$$

where the expansion coefficient $\Sigma_{s\ell}(u)$ is given by equation (36) as

$$\Sigma_{s\ell}(u) = \Sigma_s(u)\overline{P}_{\ell}(u, \mu_o). \qquad (138)$$

For $\ell = 0$, the double integral in equation (137) is the slowing down density, q_o.

A P_2 approximation to the transport equation (137) can be obtained by making the approximations

$$F_\ell = 0 \qquad \ell = 3, 4, 5, \ldots \quad . \tag{139}$$

$$F_\ell(u') = F_\ell(u) \qquad \ell = 1, 2 \qquad u - \epsilon \le u' \le u \tag{140}$$

$$\int_{u-\epsilon}^{u} \Sigma_{s\ell}(u'\to u)du' = \Sigma_{s\ell}(u) \qquad \ell = 1, 2 \tag{141}$$

and specializing to an isotropic source. Equation (141) is an excellent approximation for S wave scattering; i.e., elastic scattering which is isotropic in the center of mass system, provided that $\Sigma_s(u')$ does not vary appreciably in the interval $u - \epsilon \le u' \le u$. This fact stems from the symmetry in u and u' of the function $\Sigma_{s\ell}(u'\to u)$ if $\Sigma_s(u')$ is essentially constant and only S wave scattering is considered. In this case, equation (68) gives

$$\Sigma_{s\ell}(u'\to u) = \frac{e^{u'-u}}{1-\alpha}\,\Sigma_s(u)P_\ell(\mu_o) \ . \tag{142}$$

The cosine of the scattering angle, μ_o, is given by equation (60) as

$$\mu_o = \frac{A+1}{2}\,e^{\left(\frac{u'-u}{2}\right)} - \frac{A-1}{2}\,e^{-\left(\frac{u'-u}{2}\right)} \ .$$

With these approximations, the set of equations (137) is limited to

$$(\Sigma - \Sigma_{so})F_o + \frac{\partial q_o}{\partial u} + \frac{\partial F_1}{\partial z} = S \tag{143}$$

$$\frac{1}{3}\frac{\partial F_o}{\partial z} + (\Sigma - \Sigma_{s1})F_1 + \frac{2}{3}\frac{\partial F_2}{\partial z} = 0 \tag{144}$$

$$\frac{2}{5}\frac{\partial F_1}{\partial z} + (\Sigma - \Sigma_{s2})F_2 = 0\ . \tag{145}$$

The expression $-\frac{\partial q_o}{\partial u} + \Sigma_{so}F_o$ constitutes the slowing down source at lethargy u. Assuming the slowing down source to be a known function turns the energy dependent problem into a monoenergetic problem. The source term, S, is zero for lethargies not directly supported by a primary source.

Following the same procedure employed for the monoenergetic case, the set of equations (143), (144), and 145) can be written as

$$-D_2\frac{\partial^2 F_o}{\partial z^2} + \Sigma_a F_o = S - \frac{\partial q_o}{\partial u} - C\frac{\partial^2}{\partial z^2}\quad S - \frac{\partial q_o}{\partial u} \tag{146}$$

$$F_1 = -D_2\frac{\partial F_o}{\partial z} + C\frac{\partial}{\partial z}\quad S - \frac{\partial q_o}{\partial u} \tag{147}$$

$$F_2 = \frac{2}{5}\frac{\Sigma_a}{\Sigma_c}F_o - \frac{2}{5\Sigma_c}\quad S - \frac{\partial q_o}{\partial u}\quad . \tag{148}$$

For S wave scattering, the slowing down density obtained from Appendix B is

$$q_o(z,u) = \sum_{n=0} (-1)^n\, \xi_{on}^{(0)}\frac{\partial^n X_o}{\partial u^n}(z,u) \tag{149}$$

where

$$\xi_{on}^{(0)} = \frac{1}{(n+1)!}\ <(u-u')^{n+1}>^{(0)}$$

$$= \frac{1}{(n+1)!}\ <\ln^{n+1}\frac{E'}{E}>^{(0)}\quad . \tag{150}$$

The superscript (0) indicates that only the zero order term in $P_\ell(\mu)$ is to be considered in evaluating the ξ_{on}.

For a sufficiently small range of ϵ, the scattering collision density, $\chi_o(z,u')$, can be approximated by the first two terms of a Taylor expansion about u' = u. In this case, $q_o(z,u)$ can be approximated by the first two terms of the series in equation (149), i. e.,

$$q_o(z,u) = \xi_{oo}^{(0)} \chi_o(z,u) - \xi_{o1}^{(0)} \frac{\partial \chi_o}{\partial u}(z,u). \tag{151}$$

From equation (150),

$$\xi_{oo}^{(0)} = \langle \ln \frac{E'}{E} \rangle^{(0)} \tag{152}$$

and

$$\xi_{01}^{(0)} = \frac{1}{2} \langle \ln^2 \frac{E'}{E} \rangle^{(0)}. \tag{153}$$

Consequently, the coefficients $\xi_{00}^{(0)}$ and $\xi_{01}^{(0)}$ in equation (151) can be interpreted physically as the average logarithmic energy decrement and one half the average square of the logarithmic energy decrement respectively for the S wave approximation. From Appendix C,

$$\xi_{00}^{(0)} = \frac{1 - \alpha + \alpha \ln \alpha}{1 - \alpha} \tag{154}$$

and

$$\xi_{01}^{(0)} = \frac{1 - \alpha(1 + \epsilon + \frac{1}{2}\ \epsilon^2)}{1 - \alpha}. \tag{155}$$

It follows from equation (151) that

$$\frac{\partial q_o}{\partial u} = \xi_{00}^{(0)} \frac{\partial \chi_o}{\partial u} \tag{156}$$

to the same order as equation (151). Therefore, equation (151) becomes

$$-\xi_{01}^{(0)} \frac{\partial q_o}{\partial u} - \xi_{00}^{(0)} q_o + \xi_{00}^{(0)2} \chi_o = 0. \tag{157}$$

Solving equation (157) for $\frac{\partial q_o}{\partial u}$, substituting the result in equation (146) and noting that $\chi_o = \Sigma_s \Phi$ yields

$$-\left(D_2 + C \frac{\xi_{00}^{(0)2}}{\xi_{01}^{(0)}}\right) \frac{\partial^2 \Phi}{\partial z^2} + \left(\Sigma_a + \frac{\xi_{00}^{(0)2}}{\xi_{01}^{(0)}} \Sigma_s\right) \Phi$$

$$- \frac{\xi_{00}^{(0)}}{\xi_{01}^{(0)}} q_o = S - C \frac{\partial^2 S}{\partial z^2} - C \frac{\xi_{00}^{(0)}}{\xi_{01}^{(0)}} \frac{\partial^2 q_o}{\partial z^2} . \tag{158}$$

Equation (151) can be differentiated twice with respect to Z to obtain

$$\frac{\partial^2 q_o}{\partial z^2} = \xi_{00}^{(0)} \Sigma_s \frac{\partial^2 \Phi}{\partial z^2} - \xi_{01}^{(0)} \frac{\partial}{\partial u} \Sigma_s \frac{\partial^2 \Phi}{\partial z^2}. \tag{159}$$

Therefore, equation (158) becomes

$$-\left(D_2 + C\, \xi_{00}^{(0)} \frac{\partial}{\partial u} \Sigma_s\right) \frac{\partial^2 \Phi}{\partial z^2} + \left(\Sigma_a + \frac{\xi_{00}^{(0)2}}{\xi_{01}^{(0)}} \Sigma_s\right) \Phi$$

$$- \frac{\xi_{00}^{(0)}}{\xi_{01}^{(0)}} q_o = S - C \frac{\partial^2 S}{\partial z^2} . \tag{160}$$

Treating the term $\frac{\partial^2 \Phi}{\partial z^2}$ as a perturbation suggests the use of an iteration procedure. The validity of this procedure must at least be checked a posteriori, by evaluating $\frac{\partial^2 \Phi}{\partial z^2}$ using the approximate solution for Φ. A validity criterion for the use of this perturbation method will be derived in the last section of this chapter where the limitations of the improved age theory are discussed.

The first order approximation to equation (160) leads to the Fermi age equation with absorption. The zero order approximation yields

$$\Phi = \frac{q_o}{\xi_{00}^{(0)} \Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}} \Sigma_a} + \frac{S - C \frac{\partial^2 S}{\partial z^2}}{\Sigma_a + \frac{\xi_{00}^{(0)2}}{\xi_{01}^{(0)}} \Sigma_s} . \tag{161}$$

The second term on the right contains as $C \frac{\partial^2 S}{\partial z^2}$ the contribution purely due to the P_2 approximation. The remaining part of Φ has been known for a long time but apparently without the realization of the physical significance of the factor $\xi_{01}^{(0)}$. Furthermore, it is by no means emphasized in the literature that one is here essentially dealing with a perturbation procedure.

Substituting Φ from equation (161) into equation (146) gives

$$\frac{D_2}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a} \frac{\partial^2 q_o}{\partial z^2} - \frac{\Sigma_a}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a} q_o - \frac{\partial q_o}{\partial u} + C \frac{\partial^2}{\partial z^2} \quad \frac{\partial q_o}{\partial u}$$

$$= - \frac{\xi_{00}^{(0)}\Sigma_s}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a} \left[\quad S - \left(C - \frac{D_2}{\Sigma_s} \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)^2}} \right) \frac{\partial^2 S}{\partial z^2} - \frac{D_2 C}{\Sigma_s} \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)^2}} \frac{\partial^4 S}{\partial z^4} \right]. \quad (162)$$

From equation (157),

$$\frac{\partial q_o}{\partial u} = - \frac{\xi_{00}^{(0)}}{\xi_{01}^{(0)}} q_o + \frac{\xi_{00}^{(0)^2}}{\xi_{01}^{(0)}} \Sigma_s \Phi$$

$$= - \frac{\xi_{00}^{(0)}}{\xi_{01}^{(0)}} q_o + \frac{\xi_{00}^{(0)^2}}{\xi_{01}^{(0)}} \Sigma_s \frac{q_o}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a}$$

$$+ \frac{\xi_{00}^{(0)^2}}{\xi_{01}^{(0)}} \Sigma_s \frac{S - C \frac{\partial^2 S}{\partial z^2}}{\Sigma_a + \frac{\xi_{00}^{(0)^2}}{\xi_{01}^{(0)}}\Sigma_s}$$

$$= - \frac{\Sigma_a}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a} q_o$$

$$+ \frac{\xi_{00}^{(0)}\Sigma_s}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a} \quad S - C \frac{\partial^2 S}{\partial z^2} . \quad (163)$$

Therefore,

$$\frac{\partial^2}{\partial z^2}\ \frac{\partial q_o}{\partial u} = -\frac{\Sigma_a}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a}\frac{\partial^2 q_o}{\partial z^2} + \frac{\xi_{00}^{(0)}\Sigma_s}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a}\ \frac{\partial^2 s}{\partial z^2} - C\frac{\partial^4 s}{\partial z^4}\,. \tag{164}$$

Equation (162) can, then, be written as

$$\frac{D_2 - C\Sigma_a}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a}\frac{\partial^2 q_o}{\partial z^2} - \frac{\Sigma_a}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a}\,q_o - \frac{\partial q_o}{\partial u}$$

$$= -\frac{\xi_{00}^{(0)}\Sigma_s}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a}\ s + \frac{D_2}{\Sigma_s}\frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)2}}\frac{\partial^2 s}{\partial z^2} - \frac{D_2 C}{\Sigma_s}\frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)2}} + C^2\frac{\partial^4 s}{\partial z^4}\,. \tag{165}$$

Defining the age with absorption

$$\tau^{(0)} = \int_0^u \frac{D_2 - C\Sigma_a}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a}\,du'\ , \tag{166}$$

the slowing down density without absorption

$$q = q_o\ \exp\int_0^u \frac{\Sigma_a}{\xi_{00}^{(0)}\Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}}\Sigma_a}\,du'\ , \tag{167}$$

and generalizing to three dimensions allows equation (165) to be recast in the familiar form of the Fermi age equation

$$\nabla^2 q = \frac{\partial q}{\partial \tau^{(0)}} - S' \tag{168}$$

where

$$S' = \frac{\xi_{00}^{(0)} \Sigma_s}{p^{(0)} (D_2 - C\Sigma_a)} \quad S + \frac{D_2}{\Sigma_s} \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)2}} \frac{\partial^2 S}{\partial z^2} - \frac{D_2 C}{\Sigma_s} \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)2}} + C^2 \quad \frac{\partial^4 S}{\partial z^4} . \tag{169}$$

The quantity $p^{(0)}$ is the resonance escape probability for the S wave scattering case defined by

$$p^{(0)} = \exp \quad -\int_0^u \frac{\Sigma_a}{\xi_{00}^{(0)} \Sigma_s + \frac{\xi_{01}^{(0)}}{\xi_{00}^{(0)}} \Sigma_a} du' . \tag{170}$$

The actual slowing down density, q_o, can be derived from q by the relation

$$q_o = pq . \tag{171}$$

The one dimensional many velocity transport equation (136) and the subsequent manipulations leading to the set of equations (146), (147), and (148), are valid within the accuracy of the approximations (139), (140), and (141) as long as the energy range considered does not include the thermal energy range.

Age Theory Including S, P, and D Wave Scattering.--A more accurate age equation than equation (168) can be obtained from the energy dependent transport equation (137) by consistently using P_1 approximations. This procedure involves P wave scattering; i.e., first order anisotropic scattering in the center of mass system.

The usual procedure for obtaining the age approximation to transport theory involves the assumption that $F_\ell(u') = F_\ell(u)$ for $\ell = 1$ or $\ell = 1,2$ if $u - \epsilon \leq u' \leq u$. It is then further assumed that

$$\int_{u-\epsilon}^{u} \Sigma_{s\ell}(u' \to u)\, du' = \Sigma_{s\ell}(u)$$

for $\ell = 1$ or $\ell = 1,2$. For elastic scattering which is isotropic in the center of mass system, this latter assumption is valid if $\Sigma_s(u')$ is a slowly varying function in the interval $u - \epsilon \leq u' \leq u$.

For a consistent P_1 approximation, the $\Sigma_{s\ell}(u' \to u)$ given by equation (68) as

$$\Sigma_{s\ell}(u' \to u) = \frac{e^{u'-u}}{1-\alpha} P_\ell(\mu_o)\Sigma_s(u') \sum_{n=o}^{\infty} (2n + 1)\overline{P}_n(u',\mu)P_n(\mu)$$

are considered for $\ell = 0, 1$. In addition, the first two terms of the series in $P_n(\mu)$ occurring in equation (68) are used; i.e., the isotropy condition has been relaxed. Since μ_o and μ are symmetric functions of u' and u, the $\Sigma_{s\ell}(u' \to u)$ are symmetric in u' and u except for Σ_s and $\overline{P}_n$ which are functions only of the initial lethargy, u'. Consequently, the consistent use of P_1 approximations allows only very small variations in $\overline{P}_1(u',\mu)$ as well as in $\Sigma_s(u')$ in the interval $[u - \epsilon, u]$.

A P_1 approximation to the transport equation (137) is obtained by assuming furthermore that

$$F_\ell = 0 \qquad \ell = 2, 3, 4, ---- . \tag{172}$$

and

$$F_\ell(u') = F_\ell(u) \qquad \ell = 1 \qquad u - \epsilon \le u' \le u . \tag{173}$$

The set of equations (137), under these conditions, becomes

$$\Sigma_a F_o + \frac{\partial q_o}{\partial u} + \frac{\partial F_1}{\partial z} = S \tag{174}$$

$$\frac{1}{3}\frac{\partial F_o}{\partial z} + \Sigma_b F_1 + \frac{\partial q_1}{\partial u} = 0. \tag{175}$$

The quantity q_1, which can be interpreted physically as the slowing down current density in the z direction, is defined by

$$q_1 \equiv \int_{u-\epsilon}^{u} du' \int_{u}^{u'+\epsilon} du'' \Sigma_{s1}(u' \to u'') F_1(z,u'). \tag{176}$$

From equation (68), in the P_1 approximation,

$$\Sigma_{s1}(u' \to u'') = \frac{e^{u'-u''}}{1 - \alpha} \mu_o \Sigma_s(u') \; [1 + 3\,\bar{\mu}\,\mu]. \tag{177}$$

If the scattering collision density, $\chi_o(z,u')$, is approximated by the first two terms of a Taylor expansion about $u' = u$, then from Appendix B,

$$q_o(z,u) = \xi_{00}^{(1)} \chi_o(z,u) - \xi_{01}^{(1)} \frac{\partial \chi_o}{\partial u}(z,u) \tag{178}$$

where

$$\xi_{00}^{(1)} = \left\langle \ln \frac{E'}{E} \right\rangle^{(1)} \tag{179}$$

and

$$\xi_{01}^{(1)} = \frac{1}{2} \left\langle \ln^2 \frac{E'}{E} \right\rangle^{(1)}. \tag{180}$$

The superscript (1) indicates that the first two terms in $P_\ell(\mu)$ are considered. The coefficients $\xi_{00}^{(1)}$ and $\xi_{01}^{(1)}$ represent the P_1 approximations to the average logarithmic energy decrement and one half the average of the squared logarithmic energy decrement respectively.

Similarly, if $\chi_1(z,u') \equiv \Sigma_s(u')F_1(z,u')$ is approximated by the first term of a Taylor series about $u' = u$, Appendix B gives

$$q_1(z,u) = \xi_{10}^{(1)} \chi_1(z,u). \tag{181}$$

The newly introduced quantity

$$\xi_{10}^{(1)} = \left\langle \mu_o \ln \frac{E'}{E} \right\rangle^{(1)}$$

represents the P_1 approximation to the average product of the cosine of the scattering angle in the laboratory system with the logarithmic energy decrement.

Differentiating equations (178) and (181) with respect to u yields

$$\frac{\partial q_o}{\partial u} = \xi_{00}^{(1)} \frac{\partial \chi_o}{\partial u} + \chi_o \frac{d\xi_{00}^{(1)}}{du} - \frac{\partial \chi_o}{\partial u} \frac{d\xi_{01}^{(1)}}{du} \tag{182}$$

and

$$\frac{\partial q_1}{\partial u} = \chi_1 \frac{d\xi_{10}^{(1)}}{du} \tag{183}$$

to the same order in $\frac{\partial}{\partial u}$ as equations (178) and (181). From equation (182),

$$\frac{\partial \chi_o}{\partial u} = \frac{1}{\xi_{00}^{(1)} - \frac{d\xi_{01}^{(1)}}{du}} \left[\frac{\partial q_o}{\partial u} - \chi_o \frac{d\xi_{00}^{(1)}}{du} \right]. \tag{184}$$

Substituting $\frac{\partial \chi_o}{\partial u}$ from equation (184) into equation (178) leads to

$$\frac{\partial q_o}{\partial u} = \frac{1}{\xi_{01}^{(1)}} \left(\xi_{00}^{(1)^2} - \xi_{00}^{(1)} \frac{d\xi_{01}^{(1)}}{du} + \xi_{01}^{(1)} \frac{d\xi_{00}^{(1)}}{du} \right) \chi_o - \frac{1}{\xi_{01}^{(1)}} \left(\xi_{00}^{(1)} - \frac{d\xi_{01}^{(1)}}{du} \right) q_o. \tag{185}$$

It can easily be shown that equations (174) and (175) are equivalent to

$$-D_1 \frac{\partial^2 \Phi}{\partial z^2} + \Sigma_a \Phi + \frac{\partial q_o}{\partial u} - \frac{1}{\Sigma_b} \frac{\partial}{\partial z} \frac{\partial q_1}{\partial u} = S \tag{186}$$

and

$$F_1 = -D_1 \frac{\partial \Phi}{\partial z} - \frac{1}{\Sigma_b} \frac{\partial q_1}{\partial u} \tag{187}$$

where

$$D_1 \equiv \frac{1}{3\Sigma_b} \tag{188}$$

is the diffusion coefficient in the P_1 approximation.

In order to obtain an equation in q_o which has the form of the Fermi age equation, it is first necessary to eliminate q_1 from equation (186). Differentiating equation (183) with respect to z yields

$$\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u} = \Sigma_s \frac{d\xi_{10}^{(1)}}{du}\frac{\partial F_1}{\partial z} . \tag{189}$$

From equation (182),

$$\frac{\partial F_1}{\partial z} = -D_1 \frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{\Sigma_b}\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u} . \tag{190}$$

Substituting $\frac{\partial F_1}{\partial z}$ from equation (190) into equation (189) and solving for $\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u}$ gives

$$\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u} = -\frac{D_1\Sigma_s \frac{d\xi_{10}^{(1)}}{du}}{1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(1)}}{du}}\frac{\partial^2 \Phi}{\partial z^2} . \tag{191}$$

Therefore, equation (186) becomes

$$\frac{D_1}{1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(1)}}{du}}\frac{\partial^2 \Phi}{\partial z^2} - \Sigma_a \Phi - \frac{\partial q_o}{\partial u} = - S . \tag{192}$$

Equation (192) differs from conventional age equations only in the form of the effective diffusion coefficient, $\dfrac{D_1}{1 + \dfrac{\Sigma_s}{\Sigma_b}\dfrac{d\xi_{10}^{(1)}}{du}}$. This equation,

with the aid of equation (185), can be written as

$$\frac{D_1}{1 + \dfrac{\Sigma_s}{\Sigma_b}\dfrac{d\xi_{10}^{(1)}}{du}} \frac{\partial^2 \Phi}{\partial z^2} - \left[\Sigma_a + \frac{\Sigma_s}{\xi_{01}^{(1)}}\left(\xi_{00}^{(1)2} - \xi_{00}^{(1)} \frac{d\xi_{01}^{(1)}}{du} + \xi_{01}^{(1)} \frac{d\xi_{00}^{(1)}}{du} \right) \right] \Phi \tag{193}$$

$$+ \frac{1}{\xi_{01}^{(1)}} \left(\xi_{00}^{(1)} - \frac{d\xi_{01}^{(1)}}{du} \right) q_o = - S.$$

If the term $\dfrac{\partial^2 \Phi}{\partial z^2}$ in equation (193) is treated as a perturbation, the zero order approximation gives

$$\Phi = \frac{\left(1 - \dfrac{1}{\xi_{00}^{(1)}} \dfrac{d\xi_{01}^{(1)}}{du} \right) q_o + \dfrac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} S}{\dfrac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \Sigma_a + \left(\xi_{00}^{(1)} - \dfrac{d\xi_{01}^{(1)}}{du} + \dfrac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \dfrac{d\xi_{00}^{(1)}}{du} \right) \Sigma_s} . \tag{194}$$

Substituting Φ from equation (194) into equation (192) yields

$$\frac{D_1 \left(1 - \dfrac{1}{\xi_{00}^{(1)}} \dfrac{d\xi_{01}^{(1)}}{du} \right)}{\left[1 + \dfrac{\Sigma_s}{\Sigma_b} \dfrac{d\xi_{10}^{(1)}}{du} \right] \left[\dfrac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \Sigma_a + \left(\xi_{00}^{(1)} - \dfrac{d\xi_{01}^{(1)}}{du} + \dfrac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \dfrac{d\xi_{00}^{(1)}}{du} \right) \Sigma_s \right]} \frac{\partial^2 q_o}{\partial z^2}$$

$$- \frac{\Sigma_a \left(1 - \frac{1}{\xi_{00}^{(1)}} \frac{d\xi_{01}^{(1)}}{du} \right) q_o}{\frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \Sigma_a + \left(\xi_{00}^{(1)} - \frac{d\xi_{01}^{(1)}}{du} + \frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \frac{d\xi_{00}}{du} \right) \Sigma_s} - \frac{\partial q_o}{\partial u} \tag{195}$$

$$= - \frac{S\Sigma_s \left(\xi_{00}^{(1)} - \frac{d\xi_{01}^{(1)}}{du} + \frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \frac{d\xi_{00}^{(1)}}{du} \right) + \frac{D_1 \frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}}}{1 + \frac{\Sigma_s}{\Sigma_b} \frac{d\xi_{10}^{(1)}}{du}} \frac{\partial^2 S}{\partial z^2}}{\frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \Sigma_a + \left(\xi_{00}^{(1)} - \frac{d\xi_{01}^{(1)}}{du} + \frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \frac{d\xi_{00}^{(1)}}{du} \right) \Sigma_s} .$$

If we now define the age with absorption

$$\tau^{(1)} = \int_o^u \frac{D_1 \left(1 - \frac{1}{\xi_{00}^{(1)}} \frac{d\xi_{01}^{(1)}}{du} \right) du'}{\left[1 + \frac{\Sigma_s}{\Sigma_b} \frac{d\xi_{10}^{(1)}}{du} \right] \left[\frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \Sigma_a + \left(\xi_{00}^{(1)} - \frac{d\xi_{01}^{(1)}}{du} + \frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \frac{d\xi_{00}^{(1)}}{du} \right) \Sigma_s \right]} , \tag{196}$$

the slowing down density without absorption

$$q(z,u) = q_o(z,u) \exp \int_o^u \frac{\Sigma_a \left(1 - \frac{1}{\xi_{00}^{(1)}} \frac{d\xi_{01}^{(1)}}{du} \right) du'}{\frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \Sigma_a + \left(\xi_{00}^{(1)} - \frac{d\xi_{01}^{(1)}}{du} + \frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}} \frac{d\xi_{00}^{(1)}}{du} \right) \Sigma_s} , \tag{197}$$

and generalize equation (195) to three dimensions, the result is the Fermi age equation

$$\nabla^2 q(\vec{r},\tau) = \frac{\partial q}{\partial \tau}(\vec{r},\tau) - S' \tag{198}$$

where

$$S' = \frac{S\Sigma_s\left(1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(1)}}{du}\right)\left(\xi_{00}^{(1)} - \frac{d\xi_{01}^{(1)}}{du} + \frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}}\frac{d\xi_{00}^{(1)}}{du}\right) + D_1\frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}}\frac{\partial^2 S}{\partial z^2}}{p^{(1)}D_1\left(1 - \frac{1}{\xi_{00}^{(1)}}\frac{d\xi_{01}^{(1)}}{du}\right)} \,. \tag{199}$$

The resonance escape probability for this case is

$$p^{(1)} = \exp\left[-\int_0^u \frac{\Sigma_a\left(1 - \frac{1}{\xi_{00}^{(1)}}\frac{d\xi_{01}^{(1)}}{du}\right)du'}{\frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}}\Sigma_a + \left(\xi_{00}^{(1)} - \frac{d\xi_{01}^{(1)}}{du} + \frac{\xi_{01}^{(1)}}{\xi_{00}^{(1)}}\frac{d\xi_{00}^{(1)}}{du}\right)\Sigma_s}\right]. \tag{200}$$

The actual slowing down density, q_o, can be obtained from q by employing the relation

$$q_o = p^{(1)}q \,. \tag{201}$$

A more accurate age approximation can be obtained by employing a P_2 approximation to the energy dependent transport equation and considering first and second order scattering anisotropy, but still using only two terms in the approximation for q_o and one term for q_1. In this case, with the assumptions

$$F_\ell = 0 \qquad \ell = 3, 4, 5, \ldots , \tag{202}$$

$$F_2(u') = F_2(u), \quad u - \epsilon \leq u' \leq u \tag{203}$$

$$\int_{u-\epsilon}^{u} \Sigma_{s2}(u' \to u)du' = \Sigma_{s2}(u), \tag{204}$$

and the restriction

$$S_\ell = \begin{cases} S & \ell = 0 \\ 0 & \ell = 1, 2, 3, \ldots, \end{cases} \tag{205}$$

the set of equations (149) becomes

$$\Sigma_a F_o + \frac{\partial q_o}{\partial u} + \frac{\partial F_1}{\partial z} = S \tag{206}$$

$$\frac{1}{3}\frac{\partial F_o}{\partial z} + \Sigma_b F_1 + \frac{\partial q_1}{\partial u} + \frac{2}{3}\frac{\partial F_2}{\partial z} = 0 \tag{207}$$

$$\frac{2}{5}\frac{\partial F_1}{\partial z} + \Sigma_c F_2 = 0 . \tag{208}$$

Following a procedure analogous to that employed for the P_1 approximation leads to an age with absorption

$$\tau^{(2)} = \int_0^u \frac{\left(D_2 - C\Sigma_a\right)\left(1 - \frac{1}{\xi_{00}^{(2)}} \frac{d\xi_{01}^{(2)}}{du}\right) du'}{\left(1 + \frac{\Sigma s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}\right)\left[\frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \Sigma_a + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \frac{d\xi_{00}^{(2)}}{du}\right) \Sigma_s\right]}, \tag{209}$$

a resonance escape probability

$$p^{(2)} = \exp\left[- \int_0^u \frac{\Sigma_a \left(1 - \frac{1}{\xi_{00}^{(2)}} \frac{d\xi_{01}^{(2)}}{du}\right) du'}{\frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \Sigma_a + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \frac{d\xi_{00}^{(2)}}{du}\right) \Sigma_s} \right], \tag{210}$$

and a source term

$$S' = \frac{S\left(1 + \frac{\Sigma s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}\right)\left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \frac{d\xi_{00}^{(2)}}{du}\right)\Sigma_s + D_2 \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \frac{\partial^2 S}{\partial z^2} + \frac{C}{1 + \frac{\Sigma s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}} \left[D_2 \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} + C\Sigma_s\left(\xi_{00}^{(2)} - \frac{d\xi_{01}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \frac{d\xi_{00}^{(2)}}{du}\right)\right] \frac{\partial^4 S}{\partial z^4}}{(D_2 - C\Sigma_a)\left(1 - \frac{1}{\xi_{00}^{(2)}} \frac{d\xi_{01}^{(2)}}{du}\right) p^{(2)}} . \tag{211}$$

The coefficients $\xi_{00}^{(2)}$, $\xi_{01}^{(2)}$, and $\xi_{10}^{(2)}$ are defined in Appendix B as

$$\xi_{00}^{(2)} = \left< \ln \frac{E'}{E} \right>^{(2)}, \tag{212}$$

$$\xi_{01}^{(2)} = \frac{1}{2} < \ln^2 \frac{E'}{E} >^{(2)}, \tag{213}$$

and

$$\xi_{10}^{(2)} = < \mu_o \ln \frac{E'}{E} >^{(2)}. \tag{214}$$

The superscript (2) indicates that three terms in $P_n(\mu)$ are considered in evaluating the $\xi_{\ell m}$. The expressions (209), (210), and (211) for τ, p, and S' then include the effects of S, P, and D wave scattering. The expressions (212), (213) and (214) are evaluated in Appendix C as equations (363), (367), and (370).

A solution of the age equation based on this optimum age theory will be obtained in the next section of this chapter for the case of an infinite plane isotropic source function $S(z,u) = \delta(Z)\ \delta(u)$.

A further improvement could be made by using a P_2 approximation to the transport equation and considering, e.g., three terms in q_o, two terms in q_1, and one term in q_2. However, in this case it is not possible to obtain an equation in q which can be reduced to the form of the Fermi age equation. Equations (209), (210) and (211) then represent an optimum age theory.

Limitations of the Optimum Age Theory. -- The validity of the optimum age theory which has been developed hinges, of course, on the degree of validity of the approximations which were made in obtaining it from the transport equation. There is the matter of the representation of the angular flux by three terms;

$$F(z,u, \ ) = \frac{1}{4\pi} \left[F_o(z,u) + 3F_1(z,u)\cos\delta + \frac{5}{2} F_2(z,u) \left\{ 3\cos^2\delta - 1 \right\} \right]. \tag{215}$$

In order to obtain a validity criterion for this approximation, it is first necessary to reexamine the energy dependent transport equation (136). Using the assumptions

$$F_\ell = 0 \qquad \ell = 3, 4, 5, \ldots, \tag{202}$$

$$F_2(u') = F_2(u) \qquad u - \epsilon \leq u' \leq u \tag{203}$$

$$\int_{u-\epsilon}^{u} \Sigma_{s2}(u' \to u)du' = \Sigma_{s2}(u), \tag{204}$$

and the restriction

$$S_\ell = \begin{cases} S & \ell = 0 \\ 0 & \ell = 1, 2, 3, \ldots \end{cases} \tag{205}$$

which were employed in the development of the optimum age theory, the set of equations (137) can be written as

$$D_2 \frac{\partial^2 F_o}{\partial z^2} - \Sigma_a F_o + C \frac{\partial^2}{\partial z^2} \frac{\partial q_o}{\partial u} - \frac{\partial q_o}{\partial u} + \frac{1}{\Sigma_b} \frac{\partial}{\partial z} \frac{\partial q_1}{\partial u} = - S + C \frac{\partial^2 S}{\partial z^2} \tag{216}$$

$$F_1 = -D_2 \frac{\partial F_o}{\partial z} + C \frac{\partial}{\partial z} \left(S - \frac{\partial q_o}{\partial u} \right) - \frac{1}{\Sigma_b} \frac{\partial q_1}{\partial u} \tag{217}$$

$$F_2 = \frac{2}{5} \frac{\Sigma_a}{\Sigma_c} F_o - \frac{2}{5\Sigma_c} \left(S - \frac{\partial q_o}{\partial u} \right) . \tag{218}$$

It can be shown that the set of equations (216), (217), and (218) is equivalent to the set (206), (207), and (208).

The case in which the age theory is applied to an infinite medium which contains an isotropic infinite plane source of monoenergetic neutrons will be considered. If a primary source term

$$S(z,u) = \delta(z)\delta(u) \tag{219}$$

is utilized, equations (217) and (218) can be written as

$$F_1 = -\frac{\partial}{\partial z}\left(D_2 F_o + C\,\frac{\partial q_o}{\partial u}\right) - \frac{1}{\Sigma_b}\,\frac{\partial q_1}{\partial u} \tag{220}$$

and

$$F_2 = \frac{2}{5\Sigma_c}\left(\Sigma_a F_o + \frac{\partial q_o}{\partial u}\right) \tag{221}$$

since the region of interest here does not include the origin. It is now necessary to obtain the solution of the age equation

$$\frac{\partial^2 q}{\partial z^2}\,(z,\tau) - \frac{\partial q}{\partial \tau}\,(z,\tau) = -\,S'(z,\tau) \tag{222}$$

for the case in which $S(z,u)$ is given by equation (214) and $S'(z,u)$ by equation (211).

The source term $S(z,u)$ can be expressed in terms of $S(z,\tau)$ as

$$S(z,u) = S(z,\tau)\,\frac{d\tau}{du} \quad . \tag{223}$$

From equation (209),

$$\frac{d\tau}{du} = \frac{(D_2 - C\Sigma_a)\left(1 - \frac{1}{\xi_{00}^{(2)}}\frac{d\xi_{01}^{(2)}}{du}\right)}{\left(1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(2)}}{du}\right)\left[\frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\Sigma_a + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\frac{d\xi_{00}^{(2)}}{du}\right)\Sigma_s\right]} . \quad (224)$$

Then, since

$$S(z,\tau) = \delta(z)\delta(\tau),$$

equation (211) becomes

$$S'(z,\tau) = \frac{\begin{aligned}&\delta(z)\delta(\tau)\left(1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(2)}}{du}\right)\left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\frac{d\xi_{00}^{(2)}}{du}\right)\Sigma_s \\ &+ 2D_2\frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\frac{\delta(z)\delta(\tau)}{z^2} \\ &+ \frac{24C}{1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(2)}}{du}}\left[D_2\frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} + C\Sigma_s\left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\frac{d\xi_{00}^{(2)}}{du}\right)\right]\frac{\delta(z)\delta(\tau)}{z^4}\end{aligned}}{\left(1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(2)}}{du}\right)\left[\frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\Sigma_a + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\frac{d\xi_{00}^{(2)}}{du}\right)\Sigma_s\right]p^{(2)}} . \quad (225)$$

The Green's function for equation (222) is

$$G(z,z',\tau,\tau') = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-4(\tau-\tau')\kappa^2} \cos\,[2(z-z')\kappa]\; d\kappa \tag{226}$$

$$= \frac{1}{\sqrt{4\pi(\tau-\tau')}}\, e^{-\frac{|z-z'|^2}{4(\tau-\tau')}} \;.$$

Therefore, the solution of equation (222) is

$$q(z,\tau) = \iint \frac{1}{\sqrt{4\pi(\tau-\tau')}}\; e^{-\frac{|z-z'|^2}{4(\tau-\tau')}}\; S'(z',\tau')dz'd\tau' . \tag{227}$$

Introducing $S'(z',\tau')$ from equation (225) into equation (227) and observing that $q_o = p^{(2)}q$ leads to

$$q_o(z,\tau) = \left[\beta + \eta\left(\frac{z^2}{2\tau^2} - \frac{1}{\tau}\right) + \frac{\zeta}{\tau^3}\left(\frac{z^4}{12\tau} - z^2 + \tau\right)\right]\frac{1}{\sqrt{4\pi\tau}}\, e^{-\frac{z^2}{4\tau}} \tag{228}$$

where

$$\beta = \cfrac{1}{1 + \cfrac{\Sigma_a}{\Sigma_s\left(\cfrac{\xi_{00}^{(2)^2}}{\xi_{01}^{(2)}} - \cfrac{\xi_{00}^{(2)}}{\xi_{01}^{(2)}}\cfrac{d\xi_{01}^{(2)}}{du} + \cfrac{d\xi_{00}^{(2)}}{du}\right)}} , \tag{229}$$

$$\eta = \frac{D_2 \dfrac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}}{2\left(1 + \dfrac{\Sigma_s}{\Sigma_b}\dfrac{d\xi_{10}^{(2)}}{du}\right)\left[\dfrac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\Sigma_a + \left(\xi_{00}^{(2)} - \dfrac{d\xi_{01}^{(2)}}{du} + \dfrac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\dfrac{d\xi_{00}^{(2)}}{du}\right)\Sigma_s\right]} \tag{230}$$

and

$$\zeta = \frac{\frac{3}{4}\,C\left[D_2 \dfrac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} + C\Sigma_s\left(\xi_{00}^{(2)} - \dfrac{d\xi_{01}^{(2)}}{du} + \dfrac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\dfrac{d\xi_{00}^{(2)}}{du}\right)\right]}{\left[1 + \dfrac{\Sigma_s}{\Sigma_b}\dfrac{d\xi_{10}^{(2)}}{du}\right]^2\left[\dfrac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\Sigma_a + \left(\xi_{00}^{(2)} - \dfrac{d\xi_{01}^{(2)}}{du} + \dfrac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}}\dfrac{d\xi_{00}^{(2)}}{du}\right)\Sigma_s\right]} \tag{231}$$

if the origin is excluded. The relations

$$\int_{-\infty}^{\infty} e^{-\frac{|z-z'|^2}{4\tau}} \frac{\delta(z')}{z'^2}\, dz' = \frac{1}{4}\left[\frac{z^2}{2\tau^2} - \frac{1}{\tau}\right] e^{-\frac{z^2}{4\tau}}$$

and

$$\int_{-\infty}^{\infty} e^{-\frac{|z-z'|^2}{4\tau}} \frac{\delta(z')}{z'^4}\, dz' = \frac{1}{32\tau^3}\left[\frac{z^4}{12\tau} - z^2 + \tau\right] e^{-\frac{z^2}{4\tau}}$$

have been used in carrying out the integration indicated in equation (227).

Equation (194), which has the same form in the P_2 as in the P_1 approximation with the superscripts (1) replaced by (2), can be written, if the origin is excluded, as

$$F_o = \frac{\left(1 - \frac{1}{\xi_{00}^{(2)}} \frac{d\xi_{01}^{(2)}}{du}\right) q_o}{\frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \Sigma_a + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \frac{d\xi_{00}^{(2)}}{du}\right) \Sigma_s} \tag{232}$$

The derivative of q_o with respect to u can be written as

$$\frac{\partial q_o}{\partial u} = \frac{\partial q_o}{\partial \tau} \frac{d\tau}{du} . \tag{233}$$

Differentiating equation (228) with respect to τ yields

$$\frac{\partial q_o}{\partial \tau} = \left[\frac{z^2}{4\tau^2} - \frac{\nu}{z}\right] q_o \tag{234}$$

where

$$\nu = \frac{\beta + \frac{\eta}{\tau}\left(\frac{5z^2}{2\tau} - 3\right) + \frac{\zeta}{\tau^3}\left(\frac{3z^4}{4\tau} - 7z^2 + 5\tau\right)}{\tau\left[\beta + \frac{\eta}{\tau}\left(\frac{z^2}{2\tau} - 1\right) + \frac{\zeta}{\tau^3}\left(\frac{z^4}{12\tau} - z^2 + \tau\right)\right]} . \tag{235}$$

Introducing $\frac{\partial q_o}{\partial \tau}$ from equation (234) and $\frac{d\tau}{du}$ from equation (224) into equation (233) gives

$$\frac{\partial q_o}{\partial u} = \frac{\left(D_2 - C\Sigma_a\right)\left(1 - \frac{1}{\xi_{00}^{(2)}} \frac{d\xi_{01}^{(2)}}{du}\right)\left(\frac{z^2}{4\tau^2} - \frac{\nu}{z}\right) q_o}{\left(1 + \frac{\Sigma_s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}\right)\left[\frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \Sigma_a + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \frac{d\xi_{00}^{(2)}}{du}\right) \Sigma_s\right]} . \tag{236}$$

A criterion for the right side of equation (215) to be a valid representation of the angular flux, $F(z,u,\hat{\Omega})$, is

$$F_2 \ll F_1 \ . \tag{237}$$

Since equation (183), in the P_2 approximation, becomes

$$\frac{\partial q_1}{\partial u} = \Sigma_s F_1 \frac{d\xi_{10}^{(2)}}{du} , \tag{238}$$

equation (220) can be written as

$$F_1 = - \frac{1}{1 + \frac{\Sigma_s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}} \frac{\partial}{\partial z} \left(D_2 F_o + C \frac{\partial q_o}{\partial u} \right) \ . \tag{239}$$

Differentiating equation (228) with respect to z gives

$$\frac{\partial q_o}{\partial z} = - \left(\frac{z}{2\tau} - \omega \right) q \tag{240}$$

where

$$\omega = \frac{\frac{\eta z}{\tau^2} + \frac{\zeta}{\tau^3}\left(\frac{z^3}{3\tau} - 2z \right)}{\beta + \frac{\eta}{\tau}\left(\frac{z^2}{2\tau} - 1 \right) + \frac{\zeta}{\tau^3}\left(\frac{z^2}{12\tau} - z^2 + \tau \right)} . \tag{241}$$

Therefore, substituting F_o from equation (232) and $\frac{\partial q_o}{\partial u}$ from equation (236) into equation (239) yields

$$\left[D_2 \left(1 + \frac{\Sigma_s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}\right) \left(\frac{z}{2\tau} - \omega\right) + \frac{C}{2} (D_2 - C\Sigma_a) \left\{ \right.\right.$$

$$F_1 = \frac{\left.\left.\left(\frac{z^2}{2\tau^2} - \nu\right)\left(\frac{z}{2\tau} - \omega\right) - \frac{z}{\tau^2} + \frac{\partial\nu}{\partial z}\right\} \right] \left(1 - \frac{1}{\xi_{00}^{(2)}} \frac{d\xi_{01}^{(2)}}{du}\right) q_o}{\left(1 + \frac{\Sigma_s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}\right) \left[\frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \Sigma_a + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \frac{d\xi_{00}^{(2)}}{du}\right) \Sigma_s \right]} \quad . \qquad (242)$$

Introducing these expressions for F_o and $\dfrac{\partial q_o}{\partial u}$ into equation (221) gives

$$F_2 = \frac{\frac{2}{5\Sigma_c} \left[\Sigma_a \left(1 + \frac{\Sigma_s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}\right) + (D_2 - C\Sigma_a)\left(\frac{z^2}{4\tau^2} - \frac{\nu}{z}\right)\right] \left(1 - \frac{1}{\xi_{00}^{(2)}} \frac{d\xi_{01}^{(2)}}{du}\right) q_o}{\left(1 + \frac{\Sigma_s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}\right) \left[\frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \Sigma_a + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} + \frac{\xi_{01}^{(2)}}{\xi_{00}^{(2)}} \frac{d\xi_{00}^{(2)}}{du}\right) \Sigma_s \right]} \quad . \qquad (243)$$

The validity criterion (237) then becomes

$$\frac{2}{5\Sigma_c} \left[\Sigma_a \left(1 + \frac{\Sigma_s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}\right) + (D_2 - C\Sigma_a) \left(\frac{z^2}{4\tau^2} - \frac{\nu}{z}\right) \right]$$

$$\ll \left[D_2 \left(1 + \frac{\Sigma_s}{\Sigma_b} \frac{d\xi_{10}^{(2)}}{du}\right)\left(\frac{z}{2\tau} - \omega\right) \right. \qquad (244)$$

$$\left. + C(D_2 - C\Sigma_a) \left\{ \left(\frac{z^2}{4\tau^2} - \frac{\nu}{2}\right) \left(\frac{z}{2\tau} - \omega\right) - \frac{z}{\tau^2} + \frac{\partial\nu}{\partial z} \right\} \right] .$$

Next, there is the approximation of the scattering collision density, $\chi_o(z,u')$, by the first two terms of a Taylor expansion about u. The validity of this procedure requires that

$$\epsilon \frac{\partial\chi_o}{\partial u} \ll \chi_o \tag{245}$$

where ϵ is the maximum lethargy change per collision. The inequality (245) can also be written as

$$\epsilon \frac{\partial\chi_o}{\partial \tau}\frac{d\tau}{du} \ll \chi_o \, . \tag{246}$$

From equation (232),

$$\frac{\partial\chi_o}{\partial \tau} = \left[\frac{z^2}{4\tau^2} - \frac{\nu}{z}\right]\chi_o \, . \tag{247}$$

Therefore, the validity criterion (245) becomes

$$\frac{z^2}{2\tau^2} - \nu \, \frac{\epsilon}{2}\frac{d\tau}{du} \ll 1 \, . \tag{248}$$

It may be pointed out here that this result is not in agreement with the validity criterion obtained by Marshak et al. (11) for a P_1 approximation. This disagreement is due to a sign error by Marshak et al. who report a plus sign in front of their simpler equivalent of the term ν instead of a minus sign. The inequality (248) is fulfilled for physically significant values of z as long as

$$\frac{1}{\nu} \gg \frac{\epsilon}{2}\frac{d\tau}{du} \, . \tag{249}$$

where $\frac{d\tau}{du}$ is given by equation (224). This condition requires, essentially, that τ be a slowly varying function of u.

The improved age theory is also based on the approximation of the collision current density, $\chi_1(z,u')$, by the first term of a Taylor expansion about u. Such rapid convergence of $\chi_1(z,u')$ about u necessitates

$$\epsilon \frac{\partial \chi_1}{\partial u} \ll \chi_1 \; . \tag{250}$$

Equation (241) can be written in terms of F_o rather than q_o by employing the relation (233) to obtain

$$F_1 = \left[D_2 + \frac{C(D_2 - C\Sigma_a)\left(\frac{z^2}{4\tau^2} - \frac{\nu}{2}\right)}{p^{(2)}\left(1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(2)}}{du}\right)} \right] \frac{z}{2\tau} F_o \; . \tag{251}$$

Multiplying both sides of equation (251) by Σ_s gives

$$\chi_1 = \frac{\gamma z}{2\tau} \chi_o \tag{252}$$

where

$$\gamma \equiv D_2 + \frac{C(D_2 - C\Sigma_a)\left(\frac{z^2}{4\tau^2} - \frac{\nu}{2}\right)}{p^{(2)}\left(1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(2)}}{du}\right)} \; . \tag{253}$$

Differentiating equation (252) with respect to u yields

$$\frac{\partial \chi_1}{\partial u} = \frac{z}{2}\left[\frac{\chi_o}{\tau}\frac{\partial \gamma}{\partial u} + \frac{\gamma}{\tau}\frac{\partial \chi_o}{\partial u} - \frac{\chi_o \gamma}{\tau^2}\frac{d\tau}{du}\right] . \tag{254}$$

However,

$$\frac{\partial\chi_o}{\partial u} = \frac{\partial\chi_o}{\partial \tau}\frac{d\tau}{du} . \tag{255}$$

Introducing $\frac{\partial\chi_o}{\partial\tau}$ from equation (247) into equation (255) gives

$$\frac{\partial\chi_o}{\partial u} = \left(\frac{z^2}{4\tau^2} - \frac{\nu}{2} \right) \chi_o \frac{d\tau}{du} . \tag{256}$$

Therefore, equation (254) becomes

$$\frac{\partial\chi_1}{\partial u} = \frac{z}{2}\left[\frac{\chi_o}{\tau}\frac{\partial\gamma}{\partial u} - \frac{\chi_o\gamma}{\tau^2}\frac{d\tau}{du} + \frac{\chi_o\gamma}{\tau}\left(\frac{z^2}{4\tau^2} - \frac{\nu}{2} \quad \frac{d\tau}{du}\right)\right] . \tag{257}$$

If it is assumed that the inequality (249) is satisfied, the last two terms on the right side of equation (257) can be neglected so that

$$\frac{\partial\chi_1}{\partial u} \simeq \frac{z\chi_o}{2\tau}\frac{\partial\gamma}{\partial u} . \tag{258}$$

The validity criterion can now be written as

$$\frac{\epsilon z\chi_o}{2\tau}\frac{\partial\gamma}{\partial u} \ll \frac{z\chi_o\gamma}{2\tau}$$

or

$$\frac{\epsilon}{\gamma}\frac{\partial\gamma}{\partial u} \ll 1 . \tag{259}$$

In order to visualize the energy dependence of the condition (259), it can be expressed as

$$E\gamma \frac{\partial}{\partial E}\left(\frac{1}{\gamma}\right) \ll 1 \; . \tag{260}$$

This condition requires that γ be a slowly varying function of E.

Because of their similarity, it is expedient to treat the condition (260) simultaneously with the approximation (204) which reads

$$\int_{u-\epsilon}^{u} du' \, \Sigma_{s2}(u' \to u) = \Sigma_{s2}(u) .$$

From equation (68), in the P_2 approximation,

$$\Sigma_{s2}(u' \to u) = \frac{\Sigma_s(u')}{1 - \alpha} e^{u'-u} P_2(\mu_o) \sum_{\ell=0}^{2} (2\ell + 1)\overline{P}_\ell(u',\mu)P_\ell(\mu) . \tag{261}$$

Therefore,

$$\int_{u-\epsilon}^{u} du' \, \Sigma_{s2}(u' \to u) = \int_{u-\epsilon}^{u} \frac{du'}{1-\alpha} \Sigma_s(u')e^{u'-u} P_2(\mu_o) \sum_{\ell=0}^{2} (2\ell+1)\overline{P}_\ell(u',\mu)P_\ell(\mu) . \tag{262}$$

If it is assumed that $\overline{P}_1(u',\mu)$ and $\overline{P}_2(u',\mu)$ are essentially constant over the lethargy range $u - \epsilon \leq u' \leq u$, equation (262) becomes, after replacing u' by x,

$$\int_{u-\epsilon}^{u} du' \, \Sigma_{s2}(u' \to u) = \int_{\alpha}^{1} \frac{dx}{1-\alpha} \Sigma_s(u')P_2(\mu_o) \sum_{\ell=0}^{2} (2\ell+1)\overline{P}_\ell(u, \mu)P_\ell(\mu) . \tag{263}$$

If we now observe that

$$\frac{1}{1-\alpha}\int_{\alpha}^{1}\left[\, P_2(\mu_o) \sum_{\ell=0}^{2} (2\ell + 1)\overline{P}_{\ell}(u,\mu)P_{\ell}(\mu) \,\right] dx = \overline{P}_2(u,\mu_o)^{(2)} \tag{264}$$

and note that in the P_2 approximation

$$\Sigma_{s2}(u) = \Sigma_s(u)\overline{P}_2(u,\mu_o)^{(2)}, \tag{265}$$

it is apparent that the approximation (204) will be valid under the additional assumption that the scattering cross section, $\Sigma_s(u')$, does not vary appreciably in the lethargy range $u - \epsilon \leq u' \leq u$. The validity criterion for this condition can be expressed in terms of energy as

$$\frac{E}{\Sigma_s}\left(e^{\epsilon} - 1\right)\frac{d\Sigma_s}{dE} \ll 1 \tag{266}$$

since

$$\left.\frac{E'}{E}\right)_{max} = \frac{1}{\alpha} = e^{\epsilon} \ .$$

The inequality (266) can also be written as

$$E\lambda_s(e^{\epsilon} - 1)\frac{d}{dE}\ \frac{1}{\lambda_s} \ll 1 \tag{267}$$

where

$$\lambda_s \equiv \frac{1}{\Sigma_s} \ . \tag{268}$$

is the scattering mean free path. The validity criteria (260) and (267) now have essentially the same form.

Finally, the validity of the perturbation procedure employed in the development of the optimum age theory must be investigated. The perturbation method was applied to the P_2 analog of equation (193). This analogous equation can be shown to be

$$\frac{D_2 + C\left[\frac{d\xi_{00}^{(2)}}{du}\Sigma_s + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du}\right)\frac{\partial}{\partial u}\Sigma_s\right]}{1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(2)}}{du}}\frac{\partial^2\Phi}{\partial Z^2}$$

$$- \left[\Sigma_a + \frac{\Sigma_s}{\xi_{01}^{(2)}}\left(\xi_{00}^{(2)2} - \xi_{00}^{(2)}\frac{d\xi_{01}^{(2)}}{du} + \xi_{01}^{(2)}\frac{d\xi_{00}^{(2)}}{du}\right)\right]\Phi \tag{269}$$

$$+ \frac{1}{\xi_{01}^{(2)}}\left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du}\right)q_o = -S + \frac{C}{1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(2)}}{du}}\frac{\partial^2 S}{\partial Z^2}.$$

A validity criterion for treating the $\frac{\partial^2}{\partial Z^2}$ term as a perturbation is

$$\frac{D_2 + C\left[\frac{d\xi_{00}^{(2)}}{du}\Sigma_s + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du}\right)\frac{\partial}{\partial u}\Sigma_s\right]}{1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(2)}}{du}}\frac{\partial^2\Phi}{\partial Z^2} \tag{270}$$

$$\ll \left[\Sigma_a + \frac{\Sigma_s}{\xi_{01}^{(2)}} \left({\xi_{00}^{(2)}}^2 - \xi_{00}^{(2)} \frac{d\xi_{01}^{(2)}}{du} + \xi_{01}^{(2)} \frac{d\xi_{00}^{(2)}}{du} \right) \right] \Phi$$

where Φ now represents the zero order approximation to the flux. This condition can be rewritten in terms of the parameter Φ defined by equation (230) to obtain

$$\left\{ 1 + \frac{C}{D_2} \left[\frac{d\xi_{00}^{(2)}}{du} \Sigma_s + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} \right) \frac{\partial}{\partial u} \quad \Sigma_s \right] \right\} \frac{\partial^2 \Phi}{\partial Z^2} \ll \frac{1}{\eta} \Phi . \qquad (271)$$

It is apparent from the form of equation (232) that the condition (271) can be replaced by

$$\epsilon\eta \frac{\partial^2 q_o}{\partial Z^2} \ll q_o$$

where

$$\epsilon = 1 + \frac{C}{D_2} \left[\frac{d\xi_{00}^{(2)}}{du} \Sigma_s + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} \right) \frac{\partial}{\partial u} \Sigma_s \right] .$$

Introducing $q_o(Z,\tau)$ from equation (228) into the inequality (272) then leads to

$$\frac{\zeta}{48\tau^5} Z^6 + \frac{1}{4\tau^3} \quad \frac{\eta}{2} - \frac{\zeta}{3\epsilon\eta} - \frac{15\zeta}{6\,\tau} \quad Z^4$$

$$+ \frac{1}{2\tau} \left[\frac{\beta}{2} - \frac{1}{\epsilon} - \frac{1}{\tau} \left(3\eta - \frac{2\zeta}{\epsilon\eta} \right) + \frac{15\zeta}{2\tau^2} \right] Z^2$$

$$- \left[\frac{\beta\tau}{\epsilon\eta} + \frac{\beta}{2} - \frac{1}{\epsilon} - \frac{1}{\tau} \left(\frac{3\eta}{2} - \frac{\zeta}{\epsilon\eta} \right) + \frac{5\zeta}{2\tau^2} \right] \ll \quad 0 . \qquad (273$$

The validity criterion (273) represents a constraint on Z.

Comparison Between Optimum and Ordinary Age Theories.--An exact comparison between the slowing down densities given by the optimum and the ordinary age theories for a particular case can not be made at the present time. The experimental measurements of the angular distributions of neutron scattering cross sections which are available are inadequate for such a comparison. However, a comparison can be made between the isotropic scattering limit of the optimum age theory and the ordinary age theory. The case of an isotropic infinite plane source of one Mev neutrons in water will be considered.

The slowing down density in this case, as predicted by the optimum age theory, is given by equation (228) as

$$q_o(Z,\tau) = \left[\beta + \eta\left(\frac{Z^2}{2\tau^2} - \frac{1}{\tau} \right) + \frac{\zeta}{\tau^3} \left(\frac{Z^4}{12\tau} - Z^2 + \tau \right) \right] \frac{e^{-\frac{Z^2}{4\tau}}}{\sqrt{4\pi\tau}} .$$

The corresponding result from the ordinary age theory is given by, e.g., Glasstone and Edlund (20) as

$$q_o(Z,\tau) = \frac{e^{-\frac{Z^2}{4\tau}}}{\sqrt{4\pi\tau}} . \tag{274}$$

A comparison of the results given by the two age theories for this particular case then reduces to a comparison of the factor

$$\Lambda = \beta + \eta\left(\frac{Z^2}{2\tau^2} - \frac{1}{\tau} \right) + \frac{\zeta}{\tau^3} \left(\frac{Z^4}{12\tau} - Z^2 + \tau \right) \tag{275}$$

with unity. The comparison will be made at thermal energy. The value of τ to be used in evaluating Λ is, therefore, the thermal age of one Mev neutrons in water. The age of neutrons in water from one Mev down to the Indium resonance at 1.4 ev has been determined by Coveyou and Sullivan (21) to be 13 cm^2. The age from 1.4 ev to thermal energy has been estimated by Weinberg and Wigner (22) to be about 2 cm^2. Consequently, τ = 15 cm^2 will be used.

In order to evaluate the parameters β, η, and ζ, it is necessary to assume that the neutron scattering is isotropic in the center of mass reference frame. This restriction is imposed by the lack of experimental scattering cross section data. For isotropic scattering in the C system, the cross section data compiled by Hughes and Schwartz (23) can be used to obtain $\beta = 0.993$, $\eta = 0.0326$ cm^2, and $\zeta = 0.00436$ cm^4.

Then Λ becomes,

$$\Lambda = 0.991 + 0.723 \times 10^{-4} cm^{-2} Z^2 + 0.718 \times 10^{-8} cm^{-4} Z^4. \tag{276}$$

In the range of Z for which $e^{-\frac{Z^2}{4\tau}}$ differs from zero by more than 0.01, the quantity Λ has a maximum deviation from unity of -0.9% and an average deviation of -0.2%.

Similar calculations for graphite and beryllium yield maximum deviations of less than -0.1%.

Therefore, in the isotropic scattering limit, the slowing down density predicted by the optimum age theory does not differ significantly from that given by the ordinary age theory. However, it is apparent from the forms of β, η, and ζ that if anisotropic scattering is considered, Λ may exhibit appreciable deviations from unity in some cases.

CHAPTER VI

EXTENSIONS BEYOND AGE TYPE THEORIES

The approximations which were made in developing the improved age theory are the highest order approximations which will yield an age theory. A case will now be examined in which the neutron slowing down process is described by an equation which does not have the form of the usual age equation. A P_2 approximation to the transport equation involving the slowing down functions q_o, q_1, and q_2 will be considered. The collision densities χ_o, χ_1 and χ_2 will be approximated by three, two, and one term of a Taylor series respectively. Second order scattering anisotropy will be considered. Since higher order approximations are involved, the slowing down density obtained will be more accurate than that given by the improved age theory.

A P_2 approximation to the transport equation (137) can be obtained by making the approximations

$$F_\ell = 0 \qquad \ell = 3, 4, 5, \ldots. \tag{277}$$

and

$$S_\ell = 0 \qquad \ell = 0, 1, 2, \ldots. \tag{278}$$

Equation (278) represents a rather stringent condition, but it will be employed for simplicity. With the assumptions (277) and (278), the set of equations (137) is restricted to

$$\Sigma_a F_o + \frac{\partial q_o}{\partial u} + \frac{\partial F_1}{\partial z} = 0 \tag{279}$$

$$\frac{1}{3}\frac{\partial F_o}{\partial z} + \Sigma_b F_1 + \frac{\partial q_1}{\partial u} + \frac{2}{3}\frac{\partial F_2}{\partial z} = 0 \tag{280}$$

$$\frac{2}{5}\frac{\partial F_1}{\partial z} + \Sigma_c F_2 + \frac{\partial q_2}{\partial u} = 0 . \tag{281}$$

where the slowing down function q_2 is defined by

$$q_2 = \int_{u-\epsilon}^{u} du' \int_{u}^{u'+\epsilon} du'' \; \Sigma_{s2}(u' \to u'')F_2(z,u') . \tag{282}$$

The set of equations (279), (280), and (281) is equivalent to the set

$$D_2 \frac{\partial^2 F_o}{\partial z^2} - \Sigma_a F_o + c \frac{\partial^2}{\partial z^2}\frac{\partial q_o}{\partial u} - \frac{\partial q_o}{\partial u} + \frac{1}{\Sigma_b}\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u} - \frac{5C}{2}\frac{\partial^2}{\partial z^2}\frac{\partial q_2}{\partial u} = 0 \tag{283}$$

$$F_1 = -D_2 \frac{\partial F_o}{\partial z} - C \frac{\partial}{\partial z}\frac{\partial q_o}{\partial u} - \frac{1}{\Sigma_b}\frac{\partial q_1}{\partial u} + \frac{5C}{2}\frac{\partial}{\partial z}\frac{\partial q_2}{\partial u} \tag{284}$$

$$F_2 = \frac{2}{5}\frac{\Sigma_a}{\Sigma_c} F_o + \frac{2}{5\Sigma_c}\frac{\partial q_o}{\partial u} - \frac{1}{\Sigma_c}\frac{\partial q_2}{\partial u} . \tag{285}$$

If the approximations

$$\chi_o(z,u') = \chi_o(z,u) + (u' - u)\frac{\partial \chi_o}{\partial u}(z,u) + \frac{(u' - u)^2}{2}\frac{\partial^2 \chi_o}{\partial u^2}(z,u), \tag{286}$$

$$\chi_1(z,u') = \chi_1(z,u) + (u' - u)\frac{\partial \chi_1}{\partial u}(z,u), \tag{287}$$

$$\chi_2(z,u') = \chi_2(z,u) \tag{288}$$

are made, Appendix B gives

$$q_o = \xi_{00}^{(2)} \chi_o - \xi_{01}^{(2)} \frac{\partial \chi_o}{\partial u} + \xi_{02}^{(2)} \frac{\partial^2 \chi_o}{\partial u^2} \quad , \tag{289}$$

$$q_1 = \xi_{10}^{(2)} \chi_1 - \xi_{11}^{(2)} \frac{\partial \chi_1}{\partial u} , \tag{290}$$

and

$$q_2 = \xi_{20}^{(2)} \chi_2 \tag{291}$$

where

$$\xi_{02}^{(2)} = \frac{1}{6} < (u - u')^3 >^{(2)} , \tag{292}$$

$$\xi_{11}^{(2)} = \frac{1}{2} < \mu_o (u - u')^2 >^{(2)} , \tag{293}$$

and

$$\xi_{20}^{(2)} = < P_2(\mu_o)(u - u') >^{(2)} . \tag{294}$$

Differentiating equations (289), (290), and (291) with respect to u gives

$$\frac{\partial q_o}{\partial u} = \chi_o \frac{d\xi_{00}^{(2)}}{du} + \xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} \quad \frac{\partial \chi_o}{\partial u} + \frac{d\xi_{02}^{(2)}}{du} - \xi_{01}^{(2)} \quad \frac{\partial^2 \chi_o}{\partial u^2} , \tag{295}$$

$$\frac{\partial q_1}{\partial u} = \chi_1 \frac{d\xi_{10}^{(2)}}{du} + \left(\xi_{10}^{(2)} - \frac{d\xi_{11}^{(2)}}{du}\right) \frac{\partial \chi_1}{\partial u} \tag{296}$$

and

$$\frac{\partial q_2}{\partial u} = \chi_2 \frac{d\xi_{20}^{(2)}}{du} \tag{297}$$

to the same orders in $\frac{\partial}{\partial u}$ as equations (289), (290), and (291).

Substituting $\frac{\partial q_2}{\partial u}$ from equation (297) into equation (285) gives

$$F_2 = \frac{2}{5} \frac{\Sigma_a}{\Sigma_c} F_o + \frac{2}{5\Sigma_c} \frac{\partial q_o}{\partial u} - \frac{\Sigma_s}{\Sigma_c} \frac{d\xi_{20}^{(2)}}{du} F_2 \, .$$

Therefore,

$$F_2 = \frac{2}{5\left(\Sigma_c + \Sigma_s \frac{d\xi_{20}^{(2)}}{du}\right)} \quad \Sigma_a F_o + \frac{\partial q_o}{\partial u} \quad . \tag{298}$$

Introducing $\frac{\partial q_2}{\partial u}$ from equation (297) into equation (284) yields

$$F_1 = D_2 \frac{\partial F_o}{\partial z} - C \frac{\partial}{\partial z} \frac{\partial q_o}{\partial u} - \frac{1}{\Sigma_b} \frac{\partial q_1}{\partial u} + \frac{5C}{2} \Sigma_s \frac{d\xi_{20}^{(2)}}{du} \frac{\partial F_2}{\partial z} \, . \tag{299}$$

Substituting F_2 from equation (298) into equation (299) leads to

$$F_1 = - \left(D_2 - \frac{C\Sigma_a \Sigma_s \frac{d\xi_{20}^{(2)}}{du}}{\Sigma_c + \Sigma_s \frac{d\xi_{20}^{(2)}}{du}} \right) \frac{\partial F_o}{\partial z} - \frac{C\Sigma_c}{\Sigma_c + \Sigma_s \frac{d\xi_{20}^{(2)}}{du}} \frac{\partial}{\partial z} \frac{\partial q_o}{\partial u} - \frac{1}{\Sigma_b} \frac{\partial q_1}{\partial u} \, . \tag{300}$$

If, for convenience, the quantities

$$B \equiv D_2 - \frac{C\Sigma_a\Sigma_s \frac{d\xi_{20}^{(2)}}{du}}{\Sigma_c + \Sigma_s \frac{d\xi_{20}^{(2)}}{du}} \tag{301}$$

and

$$H \equiv \frac{C\Sigma_c}{\Sigma_c + \Sigma_s \frac{d\xi_{20}^{(2)}}{du}} \tag{302}$$

are introduced, equation (300) becomes

$$F_1 = -B \frac{\partial F_o}{\partial z} - H \frac{\partial}{\partial z} \frac{\partial q_o}{\partial u} - \frac{1}{\Sigma_b} \frac{\partial q_1}{\partial u} . \tag{303}$$

With the aid of equations (297) and (298), equation (283) can be written as

$$D_2 \frac{\partial^2 F_o}{\partial z^2} - \Sigma_a F_o + C \frac{\partial^2}{\partial z^2} \frac{\partial q_o}{\partial u} - \frac{\partial q_o}{\partial u} + \frac{1}{\Sigma_b} \frac{\partial}{\partial z} \frac{\partial q_1}{\partial u}$$

$$- \frac{C\Sigma_s \frac{d\xi_{20}^{(2)}}{du}}{\Sigma_c + \Sigma_s \frac{d\xi_{20}^{(2)}}{du}} \left[\Sigma_a \frac{\partial^2 F_o}{\partial z^2} + \frac{\partial^2}{\partial z^2} \frac{\partial q_o}{\partial u} \right] = 0$$

or

$$B\frac{\partial^2 F_o}{\partial z^2} - \Sigma_a F_o + H\frac{\partial^2}{\partial z^2}\frac{\partial q_o}{\partial u} - \frac{\partial q_o}{\partial u} + \frac{1}{\Sigma_b}\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u} = 0 \ . \tag{304}$$

It is now necessary to eliminate q_1 from equation (304). Differentiating equation (296) with respect to z yields

$$\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u} = \frac{d\xi_{10}^{(2)}}{du}\frac{\partial \chi_1}{\partial z} + \xi_{10}^{(2)} - \frac{d\xi_{11}^{(2)}}{du}\frac{\partial}{\partial z}\frac{\partial \chi_1}{\partial u} \ . \tag{305}$$

Equation (303) can be written as

$$\chi_1 = -B\Sigma_s\frac{\partial F_o}{\partial z} - H\Sigma_s\frac{\partial}{\partial z}\frac{\partial q_o}{\partial u} - \frac{\Sigma_s}{\Sigma_b}\frac{\partial q_1}{\partial u} \tag{306}$$

Therefore,

$$\frac{\partial \chi_1}{\partial z} = -B\Sigma_s\frac{\partial^2 F_o}{\partial z^2} - H\Sigma_s\frac{\partial^2}{\partial z^2}\frac{\partial q_o}{\partial u} - \frac{\Sigma_s}{\Sigma_b}\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u} \tag{307}$$

and equation (305) becomes

$$\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u} = \frac{1}{1 + \frac{\Sigma_s}{\Sigma_b}\frac{d\xi_{10}^{(2)}}{du}}\left[\ \xi_{10}^{(2)} - \frac{d\xi_{11}^{(2)}}{du}\frac{\partial}{\partial z}\frac{\partial\ 1}{\partial u} - B\Sigma_s\frac{d\xi_{10}^{(2)}}{du}\frac{\partial^2 F_o}{\partial z^2}\right.$$

$$\left. -H\Sigma_s\frac{d\xi_{10}^{(2)}}{du}\frac{\partial^2}{\partial z^2}\frac{\partial q_o}{\partial u}\right] . \tag{308}$$

Substituting $\dfrac{\partial q_1}{\partial u}$ from equation (296) into equation (306) gives

$$\chi_1 = -K\left[B \frac{\partial F_o}{\partial z} + H \frac{\partial}{\partial z} \frac{\partial q_o}{\partial u} + L \frac{\partial \chi_1}{\partial u} \right] \tag{309}$$

where

$$K \equiv \frac{\Sigma_b}{\dfrac{\Sigma_b}{\Sigma_s} + \dfrac{d\xi_{10}^{(2)}}{du}} \tag{310}$$

and

$$L \equiv \frac{\xi_{10}^{(2)} - \dfrac{d\xi_{11}^{(2)}}{du}}{\Sigma_b} \quad . \tag{311}$$

Equation (309) can be differentiated with respect to u to obtain

$$\frac{\partial \chi_1}{\partial u} = -\left[B \frac{\partial F_o}{\partial z} + H \frac{\partial}{\partial z} \frac{\partial q_o}{\partial u} + L \frac{\partial \chi_1}{\partial u} \right] \frac{dk}{du}$$

$$- K\left[\frac{dB}{du} \frac{\partial F_o}{\partial z} + \frac{dH}{du} \frac{\partial}{\partial z} \frac{\partial q_o}{\partial u} + \frac{dL}{du} \frac{\partial \chi_1}{\partial u} \right]$$

$$- K\left[B \frac{\partial}{\partial u} \frac{\partial F_o}{\partial z} + H \frac{\partial}{\partial z} \frac{\partial^2 q_o}{\partial u^2} \right] .$$

Then,

$$\frac{\partial \chi_1}{\partial u} = -\frac{1}{1 + L\frac{dK}{du} + K\frac{dL}{du}}\left[\left(B\frac{d}{du} + K\frac{dB}{du}\right)\frac{\partial F_o}{\partial z} + \left(H\frac{dk}{du} + K\frac{dH}{du}\right)\frac{\partial}{\partial z}\frac{\partial q_o}{\partial u}\right. \tag{312}$$

$$\left. + BK\frac{\partial}{\partial u}\frac{\partial F_o}{\partial z} + HK\frac{\partial}{\partial z}\frac{\partial^2 q_o}{\partial u^2}\right].$$

Equation (308) now becomes

$$\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u} = -\frac{K}{\Sigma_s}\left[\frac{\xi_{10}^{(2)} - \frac{d\xi_{11}^{(2)}}{du}}{1 + \frac{d}{du}(LK)}\left\{\frac{d}{du}(BK)\frac{\partial^2 F_o}{\partial z^2} + \frac{d}{du}(HK)\frac{\partial^2}{\partial z^2}\frac{\partial q_o}{\partial u}\right.\right.$$

$$\left. + BK\frac{\partial}{\partial u}\frac{\partial^2 F_o}{\partial z^2} + HK\frac{\partial^2}{\partial z^2}\frac{\partial^2 q_o}{\partial u^2}\right\}$$

$$\left. + B\Sigma_s\frac{d\xi_{10}^{(2)}}{du}\frac{\partial^2 F_o}{\partial z^2} + H\Sigma_s\frac{d\xi_{10}^{(2)}}{du}\frac{\partial^2}{\partial z^2}\frac{\partial q_o}{\partial u}\right].$$

Therefore,

$$\frac{1}{\Sigma_b}\frac{\partial}{\partial z}\frac{\partial q_1}{\partial u} = -\frac{K}{\Sigma_b\Sigma_s\left[1 + \frac{d}{du}(LK)\right]}\left[\left\{\left(\xi_{10}^{(2)} - \frac{d\xi_{11}^{(2)}}{du}\right)\frac{d}{du}(BK)\right.\right.$$

$$\left. + B\Sigma_s\frac{d\xi_{10}^{(2)}}{du}\left(1 + \frac{d}{du}LK\right)\right\}\frac{\partial^2 F_o}{\partial z^2}$$

$$+\left\{\left(\xi_{10}^{(2)}-\frac{d\xi_{11}^{(2)}}{du}\right)\frac{d}{du}(HK)+H\Sigma_s\frac{d\xi_{10}^{(2)}}{du}\left(1+\frac{d}{du}LK\right)\right\}\frac{\partial^2}{\partial z^2}\frac{\partial q_o}{\partial u}$$

$$+\left(\xi_{10}^{(2)}-\frac{d\xi_{11}^{(2)}}{du}\right)BK\frac{\partial}{\partial u}\frac{\partial^2 F_o}{\partial z^2}+\left(\xi_{10}^{(2)}-\frac{d\xi_{11}^{(2)}}{du}\right)HK\frac{\partial^2}{\partial z^2}\frac{\partial^2 q_o}{\partial u^2}\ \Bigg]$$

$$=-M\frac{\partial^2 F_o}{\partial z^2}-N\frac{\partial^2}{\partial z^2}\frac{\partial q_o}{\partial u}-U\frac{\partial}{\partial u}\frac{\partial^2 F_o}{\partial z^2}-V\frac{\partial^2}{\partial z^2}\frac{\partial^2 q_o}{\partial u^2} \tag{313}$$

where

$$M\equiv\frac{K}{\Sigma_b\Sigma_s\ [1+\frac{d}{du}(LK)]}\left[\left(\xi_{10}^{(2)}-\frac{d\xi_{11}^{(2)}}{du}\right)\frac{d}{du}(BK)+B\Sigma_s\frac{d\xi_{10}^{(2)}}{du}\left\{1+\frac{d}{du}(LK)\right\}\right], \tag{314}$$

$$N\equiv\frac{K}{\Sigma_b\Sigma_s\ [1+\frac{d}{du}(LK)]}\left[\left(\xi_{10}^{(2)}-\frac{d\xi_{11}^{(2)}}{du}\right)\frac{d}{du}(HK)+H\Sigma_s\frac{d\xi_{10}^{(2)}}{du}\left\{1+\frac{d}{du}(LK)\right\}\right], \tag{315}$$

$$U\equiv\frac{K^2B\left(\xi_{10}^{(2)}-\frac{d\xi_{11}^{(2)}}{du}\right)}{\Sigma_b\Sigma_s\ [1+\frac{d}{du}(LK)]}, \tag{316}$$

and

$$V \equiv \frac{K^2 H \left(\xi_{10}^{(2)} - \frac{d\xi_{11}^{(2)}}{du} \right)}{\Sigma_b \Sigma_s \; \left[1 + \frac{d}{du} (LK) \right]} \; . \qquad (317)$$

Substituting $\frac{1}{\Sigma_b} \frac{\partial}{\partial z} \frac{\partial q_1}{\partial u}$ from equation (313) into equation (304) yields

$$(B-M) \frac{\partial^2 F_o}{\partial Z^2} - \Sigma_a F_o + (H - N) \frac{\partial^2}{\partial z^2} \frac{\partial q_o}{\partial u} - \frac{\partial q_o}{\partial u} \qquad (318)$$

$$- U \frac{\partial}{\partial u} \frac{\partial^2 F_o}{\partial z^2} - V \frac{\partial^2}{\partial z^2} \frac{\partial^2 q_o}{\partial u^2} = 0 \; .$$

This equation can be written in terms of the slowing down density, χ_o, as

$$\frac{1}{\Sigma_s} (B - M) \frac{\partial^2 {}_o}{\partial z^2} - \frac{\Sigma_a}{\Sigma_s} \chi_o + (H - N) \frac{\partial^2}{\partial z^2} \frac{\partial q_o}{\partial u} - \frac{\partial q_o}{\partial u} \qquad (319)$$

$$- U \frac{\partial}{\partial u} \frac{1}{\Sigma_s} \frac{\partial^2 \chi_o}{\partial z^2} - V \frac{\partial^2}{\partial z^2} \frac{\partial^2 q_o}{\partial u^2} = 0 \; .$$

Substituting $\frac{\partial q_o}{\partial u}$ from equation (295) into equation (319) leads to

$$\left(a + b \frac{\partial}{\partial u} + d \frac{\partial^2}{\partial u^2} \right) \chi_o \qquad (320)$$

$$+ \left(e + f \frac{\partial}{\partial u} + h \frac{\partial^2}{\partial u^2} \right) \frac{\partial^2 \chi_o}{\partial z^2} = 0$$

where

$$a \equiv -\frac{\Sigma_a}{\Sigma_s} - \frac{d\xi_{00}^{(2)}}{du} \tag{321}$$

$$b \equiv \frac{d\xi_{01}^{(2)}}{du} - \xi_{00}^{(2)} \ , \tag{322}$$

$$d \equiv \xi_{01}^{(2)} - \frac{d\xi_{02}^{(2)}}{du} \ , \tag{323}$$

$$e \equiv \frac{1}{\Sigma_s} (B - M) + (H - N) \frac{d\xi_{00}^{(2)}}{du} - U \frac{d\lambda s}{du} - V \frac{d^2 \xi_{00}^{(2)}}{du^2} \ , \tag{324}$$

$$f \equiv (H - N) \quad \xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} - U\lambda s - V \left(2 \frac{d\xi_{00}^{(2)}}{du} - \frac{d^2 \xi_{01}^{(2)}}{du} \right), \tag{325}$$

and

$$h \equiv (H - N) \quad \frac{d\xi_{02}^{(2)}}{du} - \xi_{01}^{(2)} \quad - V \left(\xi_{00}^{(2)} - 2 \frac{d\xi_{01}^{(2)}}{du} + \frac{d^2 \xi_{02}^{(2)}}{du^2} \right) . \tag{326}$$

The slowing down density can be obtained from the simultaneous solution of equations (320) and equation (295) which reads

$$\frac{\partial q_o}{\partial u} = \chi_o \frac{d\xi_{00}^{(2)}}{du} + \left(\xi_{00}^{(2)} - \frac{d\xi_{01}^{(2)}}{du} \right) \frac{\partial \chi_o}{\partial u} + \left(\frac{d\xi^{(2)}}{du} - \xi_{01}^{(2)} \right) \frac{\partial^2 \chi_o}{\partial u^2} \ .$$

The scalar flux can be obtained directly from the solution of equation (320).

APPENDIX A

AVERAGE VALUES

The average value of any fuction $Q(\mu)$ with respect to the angular distribution of the scattering cross section is given by

$$< Q(\mu) > \quad \overline{Q}(\mu) = \frac{\int_{-1}^{1} Q(\mu) f(\mu) d}{\int_{-1}^{1} f(\mu) d\mu} \quad . \tag{327}$$

However,

$$\int_{-1}^{1} f(\mu) d\mu = 1.$$

Therefore, equation (327) becomes

$$< Q(\mu) > = \int_{-1}^{1} Q(\mu) f(\mu) d\mu. \tag{328}$$

Since the function $Q(\mu)$ could also be expressed as $Q(E', E)$ it is sometimes more convenient to average over the final energy, E, rather than μ. Equation (328) can be written in terms of energy as

$$< Q(E', E) > = \int_{E'}^{\alpha E} Q(E', E) f(E) dE. \tag{329}$$

The probability that, after scattering, a neutron with initial energy E' will have an energy in the interval E to E + dE is

$$f(E)dE = -f(\mu)\frac{d\mu}{dE}\,dE \; . \tag{330}$$

Differentiating equation (64) with respect to E gives

$$\frac{d\mu}{dE} = \frac{2}{E'(1 - \alpha)} \; . \tag{331}$$

Introducing $\frac{d\mu}{dE}$ from equation (331) and $f(\mu)$ from equation (28) into equation (330) yields

$$f(E)dE = -\left[\sum_{\ell=0}^{\infty} (2\ell + 1)\overline{P}_\ell(E',\mu)\; P_\ell(\mu)\right] \frac{dE}{E'(1 - \alpha)} \; . \tag{332}$$

Therefore, equation (329) becomes

$$< Q(E',E) > = \int_{\alpha E'}^{E'} \frac{Q(E',E)}{E'(1 - \alpha)} \sum_{\ell=0}^{\infty} (2\ell + 1)\overline{P}_\ell(E',\mu)P_\ell(\mu)dE. \tag{333}$$

The result of changing the variable of integration from E to $x = \frac{E}{E'}$ in equation (333) is

$$< Q > = \frac{1}{1 - \alpha} \int_{\alpha}^{1} Q(x) \sum_{\ell=0}^{\infty} (2\ell + 1)\overline{P}_\ell(E', \mu)P_\ell(\mu)dx \tag{334}$$

if $Q(E', E) = Q(E/E')$.

APPENDIX B

SLOWING DOWN FUNCTIONS

The slowing down functions, q_ℓ, defined by

$$q_\ell(z,u) = \int_{u-\epsilon}^{u} du' \int_{u}^{u'+E} du'' \Sigma_{s\ell}(z,u' \rightarrow u'') F_\ell(z,u') \tag{335}$$

will now be examined in detail. The quantity ϵ is defined by

$$\epsilon \equiv \ln \frac{1}{\alpha} .$$

From equation (21),

$$\left.\frac{E'}{E}\right)_{max} = \frac{1}{\alpha} \quad . \tag{336}$$

Equation (336) can be written in terms of lethargy as

$$(u - u')_{max} = \ln \frac{1}{\alpha} . \tag{337}$$

Therefore, ϵ represents the maximum lethargy change per collision.

An examination of Figure 4 shows that the integral over u'' in equation (335) gives the slowing down source term due to neutrons which are scattered from lethargy u' past lethargy u. The integral over u' represents a sum over all lethargy levels from which a neutron could attain a lethargy greater than or equal to u in a single collision.

The $\ell = 0$ term, q_0, is the ordinary slowing down density, i.e., the number of neutrons per unit volume and per unit time that slow down past a given energy E or lethargy u.

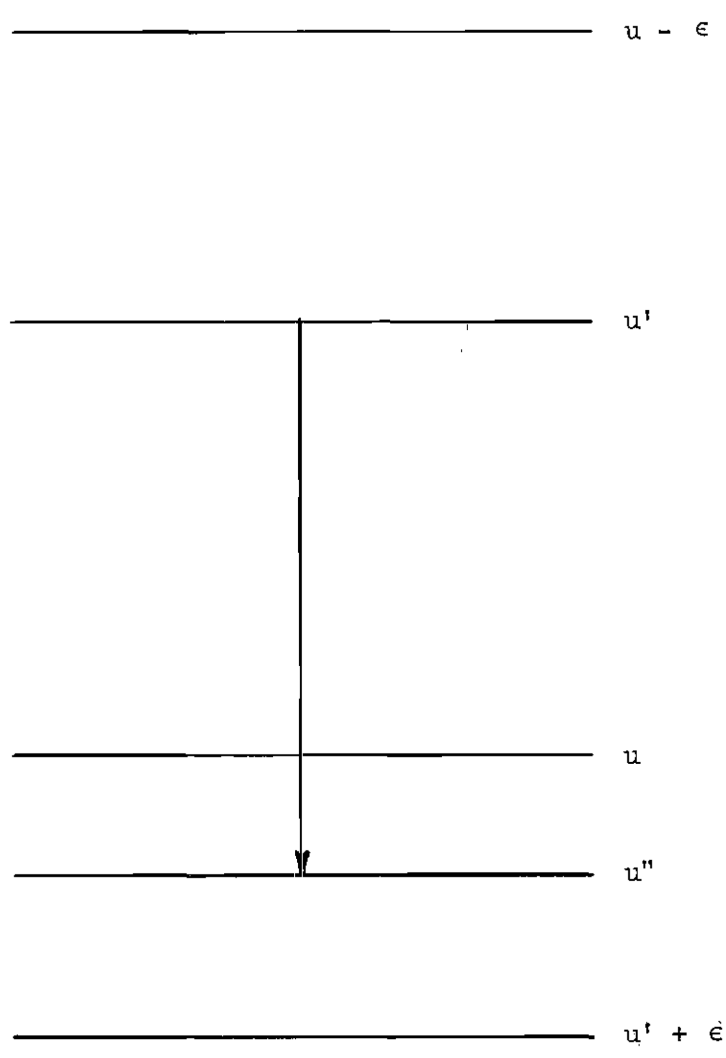


Figure 4. Lethargy Levels Involved in the Slowing Down Functions

The $\ell = 1$ term represents the slowing down current density in the z direction.

Substituting the expansion coefficient $\Sigma_{s\ell}(z,u' \rightarrow u'')$ from equation (61) into equation (335) gives

$$q_\ell(z,u) = \int_{u-\epsilon}^{u} du' \int_{u}^{u'+\epsilon} \left[\frac{\chi_\ell(z,u)}{1-\alpha} e^{u'-u''} P_\ell(\mu_o) \sum_{m=o}^{\infty} (2m+1)\overline{P}_m(u',\mu)P_m(\mu) \right] du'' \tag{338}$$

where $\chi_\ell(z,u')$ is the scattering collision function defined as

$$\chi_\ell(z,u') \quad \Sigma_s(z,u')F_\ell(z,u').$$

The scattering collision function $\chi_\ell(z,u')$ is then represented by a Taylor expansion about $u' = u$; i.e.,

$$\chi_\ell(z,u') = \sum_{n=o}^{\infty} \frac{(u' - u)^n}{n!} \frac{\partial^n \chi_\ell}{\partial u^n} (z,u). \tag{339}$$

If it is assumed that the $\overline{P}_m(u',\mu)$ do not vary appreciably in the lethargy interval $u - \epsilon \leq u' \leq u$, equation (338) becomes

$$q_\ell(z,u) = \sum_{m,n=o}^{\infty} \frac{(2m + 1)}{n!(1 - \alpha)} \overline{P}_m(u,\mu) \frac{\partial^n \chi_\ell}{\partial u^n}(z,u) \int_{u-\epsilon}^{u} (u'-u)^n du' \int_{u}^{u'+\epsilon} P_\ell(\mu_o) P_m(\mu) e^{u'-u''} du''. \tag{340}$$

It is convenient to change the variables of integration from u" and u' to

$$x \equiv e^{u'-u''} \tag{341}$$

and

$$x \equiv e^{u'-u}. \tag{342}$$

This procedure yields

$$q_{\ell}(z,u) = \sum_{m,n=0}^{\infty} \frac{(2m+1)\overline{P}_m(u,\mu)}{n!(1-\alpha)} \frac{\partial^n \chi_{\ell}}{\partial u^n}(z,u) \int_1^{\alpha} \frac{\ln^n x}{x} dx \int_x^{\alpha} P_{\ell}(\mu'_o)P_m(\mu')dx' . \tag{343}$$

Equation (54) gives

$$\mu' \equiv \mu(x') = \frac{2x' - (1+\alpha)}{1-\alpha} \tag{344}$$

and from equation (49),

$$\mu'_o \equiv \mu_o(x') = \frac{A+1}{2}(x')^{\frac{1}{2}} - \frac{A-1}{2}(x')^{-\frac{1}{2}} . \tag{345}$$

A further change of variable from x' to μ' in equation (343) gives

$$q_{\ell}(z,u) = \sum_{m,n=0}^{\infty} \frac{(2m+1)\overline{P}_m(Z,u)}{2n!} \frac{\partial^n \chi_{\ell}}{\partial u^n}(z,u) \int_1^{\alpha} \frac{\ln^n x}{x} dx \int_{\mu}^{-1} P_{\ell}(\mu'_o)P_m(\mu')d\mu' . \tag{346}$$

Equation (346) can be written as

$$q_{\ell}(z,u) = \sum_{m,n=0}^{\infty} \frac{(2m+1)\overline{P}_m(Z,\mu)}{2n!} \frac{\partial^n \chi_{\ell}}{\partial u^n}(z,u) \int_1^{\alpha} \frac{\ln^n x}{x} \Gamma(\mu)dx \tag{347}$$

where

$$\Gamma(\mu) \equiv \int_{\mu}^{1} P\ (\mu'_o)P_m(\mu')d\mu' . \tag{348}$$

Now, let

$$R = \Gamma(\mu) \tag{349}$$

and

$$dS = \frac{\ln^n x}{x}\, dx. \tag{350}$$

Then

$$dR = \frac{d\Gamma}{d\mu}\frac{d\mu}{dx}\, dx$$

$$= \frac{2dx}{1-\alpha}\frac{d}{d\mu}\int_{\mu}^{-1} P_\ell(\mu'_o)P_m(\mu')d\mu'$$

$$= -\frac{2dx}{1-\alpha} P_\ell(\mu_o)P_m(\mu) \tag{351}$$

and

$$S = \int \frac{\ln^n x}{x}\, dx$$

$$= \frac{1}{n+1}\ln^{n+1} x. \tag{352}$$

Therefore, integrating once by parts in equation (347) yields

$$q_\ell(z,u) = \sum_{m,n=o}^{\infty} \frac{(2m+1)\overline{P}_m(u,\mu)}{2n!}\frac{\partial^n \chi_\ell}{\partial u^n}(z,u)\left\{\frac{1}{n+1}\Gamma(\mu)\ln^{n+1} x\right]_1^{\alpha},$$

$$+ \frac{2}{(1-\alpha)(n+1)} \int_1^{\alpha} \ln^{n+1} x P_\ell(\mu_o) P_m(\mu) dx \quad . \tag{353}$$

The Legendre polynomials, $P_k(y)$ can be generated by the Rodrigues formula

$$P_k(y) = \frac{1}{2^k K!} \frac{d^k}{dy^k} (y^2 - 1)^k. \tag{354}$$

If this representation is employed in equation (348), it is apparent that since $\mu_o = \mu = -1$ for $x = \alpha$, we must have $\Gamma(\mu) = 0$ when $x = \alpha$. This is a consequence of the fact that

$$\left[\frac{d^i}{dy^i} (y^2 - 1)^k \right]_{y=-1} = 0$$

for $i \neq k$.

Equation (353) then reduces to

$$q_\ell(z,u) = \sum_{m,n=o}^{\infty} \frac{(2m+1)\overline{P}_m(u,\mu)}{(n+1)!(1-\alpha)} \frac{\partial^n \chi_\ell}{\partial u^n}(z,u) \int_1^{\alpha} \ln^{n+1} x P_\ell(\mu_o) P_m(\mu) dx. \tag{355}$$

It has been shown in Appendix A that

$$\frac{1}{1-\alpha} \int_{\alpha}^{1} f(x) \sum_{m=o}^{\infty} (zm + 1)\overline{P}_m(u,\mu) \; P_m(\mu) dx = < f(x) > \quad .$$

Therefore,

$$\frac{(-1)^n}{1-\alpha} \int_1^{\alpha} P\,(\mu_o) \ln^{n+1} x \sum_{m=o}^{\infty} (2m + 1)\overline{P}_m(u,\mu) P_m(\mu) dx$$

$$= < -P_\ell(\mu_o) \ln^{n+1} x >$$

$$= < P_\ell(\mu_o)\ (u - u')^{n+1} > \tag{356}$$

where the factor $(-1)^n$ has been added to insure that the average values obtained are always positive.

With the aid of equation (356), equation (355) can be written as

$$q_\ell(z,u) = \sum_{n=o}^{\infty} \frac{(-1)^n}{(n+1)!} < P_\ell(\mu_o)(u - u')^{n+1} > \frac{\partial^n \chi_\ell}{\partial u^n}(z,u) \ . \tag{357}$$

If we define

$$\xi_{\ell n} = \frac{1}{(n+1)!} < P_\ell(\mu_o)\ (u - u')^{n+1} > , \tag{358}$$

equation (357) becomes

$$q_\ell(z,u) = \sum_{n=o}^{\infty} (-1)^n \xi_{\ell\, n} \frac{\partial^n \chi_\ell}{\partial u^n}(z,u) \ . \tag{359}$$

It should be noted that equation (359) is based on the assumption that the $\overline{P}_m(u', \mu)$ are essentially constant in the lethargy interval $u - \epsilon \leq u' \leq u$. Consequently, this equation is exactly correct only for S wave scattering.

APPENDIX C

INPUT FUNCTIONS

It has been demonstrated that the anisotropy of the neutron scattering process in the center of mass reference frame can be introduced into neutron transport theory by means of certain input functions. These input function appear in nuclear reactor calculations as quantities which have been averaged over the angular distribution of the neutron scattering cross section.

The solution of a P_2 approximation to the monoenergetic transport equation involves the average cosine, $\overline{\mu}_o$, and the average squared cosine, $\overline{\mu_o^2}$, of the scattering angle in the laboratory system. These quantities appear in

$$E_b = \Sigma - \Sigma_{s1}$$

$$= \Sigma - \Sigma_s \overline{P}_1(E',\mu_o)$$

and

$$\Sigma_c = \Sigma - \Sigma_{s2}$$

$$= \Sigma - \Sigma_s \overline{P}_2(E',\mu_o) \ .$$

The optimum age theory which was developed in Chapter V contains the average logarithmic energy decrement, ξ_{oo}; one half the average square of the logarithmic energy decrement, ξ_{01}; and the average product of μ_o and the logarithmic energy decrement, ξ_{10}, as well as $\overline{P}_1(E',\mu_o)$ and $\overline{P}_2(E',\mu_o)$.

In order to apply the equations which have been obtained to particular cases, it is necessary to relate these input functions to the experimental cross section data converted to the center of mass system. Since the angular distribution of the scattering cross section varies with the initial neutron energy, the input functions are, in general, energy dependent.

The input functions $\overline{P}_1(E',\mu_o)$, $\overline{P}_2(E', \mu_o)$, ξ_{00}, ξ_{01}, and ξ_{10} will now be evaluated by using equations (328) and (334) and considering only the first three terms of the series expression for $f(\mu)$. This procedure will yield the P_2 approximation to the input functions.

Average Logarithmic Energy Decrement.--The average logarithmic energy decrement per collision is

$$\xi_{00} = \left< \ln \frac{E'}{E} \right> \quad . \tag{360}$$

The P_2 approximation to equation (334) then gives the input function $\xi_{00}^{(2)}$ as

$$\xi_{00}^{(2)} = \int_1^{\alpha} \frac{\ln x}{1 - \alpha} \sum_{\ell=o}^{\infty} (2\ell + 1)\overline{P}_{\ell}(E',\mu)P_{\ell}(\mu)dx. \tag{361}$$

Introducing

$$\mu = \frac{2x - (1 + \alpha)}{1 - \alpha}$$

into equation (361) yields

$$\xi_{00}^{(2)} = \frac{1}{1-\alpha}\int_1^\alpha \ln x \left[1 - 3\overline{P}_1(E',\mu)\left(\frac{1+\alpha}{1-\alpha}\right) + \frac{5}{2}\,\overline{P}_2(E',\mu)\left\{3\left(\frac{1+\alpha}{1-\alpha}\right)^2 - 1\right\} + \left\{\frac{6\overline{P}_1(E',\mu)}{1-\alpha} - \frac{30\overline{P}_2(E',\mu)}{(1-\alpha)^2}(1+\alpha)\right\} x + \frac{30\overline{P}_2(E',\mu)}{(1-\alpha)^2} x^2 \right] dx \,. \tag{362}$$

Carrying out the indicated integration in equation (362) leads to

$$\xi_{00}^{(2)} = 1 + \frac{\alpha\ln\alpha}{1-\alpha} - \frac{3\overline{P}_1(E',\mu)}{2(1-\alpha)^2}\left[1 - \alpha(\alpha - 2\ln\alpha)\right] + \frac{5\overline{P}_2(E',\mu)}{(1-\alpha)^3}\left[\frac{1-\alpha}{6}(1 + 10\alpha + \alpha^2) + \alpha\ln\alpha(1+\alpha)\right] \,. \tag{363}$$

The functions

$$\overline{P}_1(E',\mu) = \overline{\mu}$$

and

$$\overline{P}_2(E',\mu) = \frac{1}{2}(3\overline{\mu}^2 - 1)$$

are obtained from the experimentally determined angular distribution of the scattering cross section for the material and neutron energy under consideration.

<u>Average Squared Logarithmic Energy Decrement</u>.--The parameter ξ_{01} defined by

$$\xi_{01} \equiv \frac{1}{2} \left\langle \ln^2 \frac{E'}{E} \right\rangle$$

will be evaluated by employing the P_2 approximation to equation (334). In this case, equation (334) becomes

$$\xi_{01}^{(2)} = -\frac{1}{2}\int_1^{\alpha} \ln^2 x \sum_{\ell=0}^{2} (2\ell+1)\overline{P}_{\ell}(E',\mu)P_{\ell}(\mu)\,\frac{dx}{1-\alpha} \quad . \tag{364}$$

Introducing

$$\mu = \frac{2x-(1+\alpha)}{1-\alpha}$$

into equation (364) leads to

$$\xi_{01}^{(2)} = -\frac{1}{2(1-\alpha)}\int_1^{\alpha} \ln^2 x \Bigg[1 - 3\overline{P}_1(E',\mu)\left(\frac{1+\alpha}{1-\alpha}\right) + \frac{5}{2}\,\overline{P}_2(E',\mu)\left\{3\left(\frac{1+\alpha}{1-\alpha}\right)^2 - 1\right\}$$
$$+\left\{\frac{6\overline{P}_1(E',\mu)}{1-\alpha} - \frac{30\overline{P}_2(E',\mu)}{(1-\alpha)^2}(1+\alpha)\right\} x + \frac{30P_2(E',\mu)}{(1-\alpha)^2}\, x^2 \Bigg]\, dx \, . \tag{365}$$

The identity

$$\int x^m \ln^n x dx \equiv \frac{x^{m+1}\ln^n x}{m+1} - \frac{n}{m+1}\int x^m \ln^{n-1} x dx \tag{366}$$

can be used to transform equation (365) into a form which can be readily integrated. Performing the indicated integration and rearranging terms yields

$$\xi_{01}^{(2)} = 1 + \frac{\alpha(\ln\alpha - \frac{1}{2}\ln^2\alpha)}{1-\alpha}$$

$$- \frac{3\overline{P}_1(E',\mu)}{(1-\alpha)^2}\left[\ \alpha\left\{\ln\alpha(1+\tfrac{1}{2}\alpha) - \tfrac{1}{2}\ln^2\alpha\right\} - \tfrac{3}{4}(\alpha^2-1)\right]$$

$$+ \frac{5\overline{P}_2(E',\mu)}{2(1-\alpha)^3}\left[(\alpha-1)\alpha\ln^2\alpha + (2+\alpha+\tfrac{1}{3}\alpha^2)\alpha\ln\alpha \right. \tag{367}$$

$$\left. + \frac{1}{18}(17 + 9\alpha - 9\alpha^2 - 17\alpha^3)\right] .$$

<u>Average Product of the Scattering Angle in the Laboratory System and the Logarithmic Energy Decrement.</u>--The input parameter ξ_{10} defined by

$$\xi_{10} \quad < \mu_o \ln \frac{E'}{E} >$$

will be evaluated, in the P_2 approximation, by using equation (334). Equation (334) gives

$$\xi_{10}^{(2)} = \frac{1}{1-\alpha}\int_1^\alpha \mu_o \ln x \sum_{\ell=o}^{\infty} (2\ell+1)\overline{P}_\ell(E',\mu)P_\ell(\mu)dx$$

$$= \frac{1}{1-\alpha}\int_1^\alpha \mu_o \ln x dx + \frac{3\overline{P}_1(E',\mu)}{1-\alpha}\int_1^\alpha \mu_o \ln x \mu dx \tag{368}$$

$$+ \frac{5\overline{P}_2(E',\mu)}{2(1-\alpha)}\int_1^\alpha \mu_o \ln x(3\ {}^2 - 1)dx .$$

Introducing

$$\mu_o = \frac{A+1}{2}X^{\frac{1}{2}} - \frac{A-1}{2}X^{-\frac{1}{2}}$$

and

$$\mu = \frac{2x - (1 + \alpha)}{1 - \alpha}$$

into equation (368) yields

$$\begin{aligned}
\xi_{10}^{(2)} = {} & \frac{1}{1 - \alpha} \int_1^\alpha \frac{A + 1}{2} x^{\frac{1}{2}} \ln x dx - \frac{1}{1 - \alpha} \int_1^\alpha \frac{A - 1}{2} x^{\frac{1}{2}} \ln x dx \\
& + \frac{3\bar{P}_1(E,\mu)}{(1 - \alpha)^2} \left[(A + 1) \int_1^\alpha x^{\frac{3}{2}} \ln x dx - (A - 1) \int_1^\alpha x^{\frac{1}{2}} \ln x dx \right. \\
& \left. - \frac{(A+1)}{2} (1+\alpha) \int_1^\alpha x^{\frac{1}{2}} \ln x dx + \frac{(A - 1)}{2} (1 + \alpha) \int_1^\alpha x^{-\frac{1}{2}} \ln x dx \right] \\
& + \frac{5\bar{P}_2(E',\mu)}{2(1 - \alpha)^3} \left[6(A + 1) \int_1^\alpha x^{\frac{5}{2}} \ln x dx - 6(A - 1) \int_1^\alpha x^{\frac{3}{2}} \ln x dx \right. \\
& - 6(A+1)(1+\alpha) \int_1^\alpha x^{\frac{3}{2}} \ln x dx + 6(A-1)(1+\alpha) \int_1^\alpha x^{\frac{1}{2}} \ln x dx \\
& + (1 + 4\alpha + \alpha^2)(A + 1) \int_1^\alpha x^{\frac{1}{2}} \ln x dx \\
& \left. - (1 + 4\alpha + \alpha^2)(A - 1) \int_1^\alpha x^{-\frac{1}{2}} \ln x dx \right].
\end{aligned} \tag{369}$$

Using the relation

$$\int x^m \ln x dx \equiv \frac{x^{m+1}}{m + 1} \left(\ln x - \frac{1}{m + 1} \right)$$

and combining terms in equation (369) yields

$$\xi_{10}^{(2)} = \frac{1}{1-\alpha}\left[(A+1)\left\{\frac{1}{3}\alpha^{\frac{3}{2}}\left(\ln\alpha - \frac{2}{3}\right) + \frac{2}{9}\right\}\right.$$

$$\left. - (A-1)\left\{\alpha^{\frac{1}{2}}(\ln\alpha - 2) + 2\right\}\right]$$

$$+ \frac{3\overline{P}_1(E',\mu)}{(1-\alpha)^2}\left[(A+1)\left\{\frac{1}{15}\alpha^{\frac{5}{2}}\left(\ln\alpha + \frac{14}{15}\right) - \frac{1}{3}\alpha^{\frac{3}{2}}\left(\ln\alpha - \frac{2}{3}\right)\right.\right.$$

$$\left. - \frac{2}{9}\left(\frac{7}{25} + \alpha\right)\right\}$$

$$\left. - (A-1)\left\{\frac{1}{3}\alpha^{\frac{3}{2}}\left(\frac{14}{3} - \ln\alpha\right) - \alpha^{\frac{1}{2}}(\ln\alpha - 2) - 2\left(\frac{7}{9} + \alpha\right)\right\}\right] \tag{370}$$

$$+ \frac{5\overline{P}_2(E',\mu)}{(1-\alpha)^3}\left[(A+1)\left\{\frac{1}{105}\alpha^{\frac{7}{2}}\left(\frac{142}{105} - \ln\alpha\right) + \frac{2}{15}\alpha^{\frac{5}{2}}\left(\ln\alpha - \frac{46}{15}\right)\right.\right.$$

$$\left. + \frac{1}{3}\alpha^{\frac{3}{2}}\left(\ln\alpha - \frac{2}{3}\right) - \frac{2}{9}\left(\frac{71}{735} - \frac{46}{15}\alpha - \alpha^2\right)\right\}$$

$$- (A-1)\left\{\frac{1}{5}\alpha^{\frac{5}{2}}\left(\ln\alpha - \frac{86}{15}\right) + 2\alpha^{\frac{3}{2}}\left(\ln\alpha - \frac{10}{3}\right)\right.$$

$$\left.\left. + \alpha^{\frac{1}{2}}(\ln\alpha - 2) + 2\left(\frac{43}{75} + \frac{10}{3}\alpha + \alpha^2\right)\right\}\right].$$

<u>Average Cosine of the Scattering Angle in the Laboratory System.</u>--The average value of $\mu_o = \cos\psi$ is given by equation (328) as

$$\overline{\mu}_o = \int_{-1}^{1} \mu_o f(\mu) d\mu \ . \tag{371}$$

Introducing $f(\mu)$ from equation (28) into equation (371) yields

$$\mu_o = \int_{-1}^{1} \mu_o \sum_{\ell=o}^{\infty} \frac{2 \ \ + 1}{2} \overline{P}_\ell (E',\mu) P_\ell(\mu) d\mu \ . \tag{372}$$

The quantities μ_o and μ are related by equation (16) which reads

$$\mu_o = \frac{A\mu + 1}{A^2 + 2A\mu + 1} \ .$$

This equation can be rewritten as

$$\mu_o = \frac{\frac{1}{A} + \mu}{1 + (\frac{1}{A})^2 + \frac{2}{A}\mu} \ . \tag{373}$$

If it is observed that the generating function for an infinite series of Legendre polynomials in μ has the form

$$\frac{1}{1 + x^2 - 2x\mu} = \sum_{\ell=0}^{\infty} x \, P_\ell(\mu), \tag{374}$$

then equation (373) becomes

$$\mu_o = (\mu + \frac{1}{A}) \sum_{\ell=o}^{\infty} (- \frac{1}{A})^\ell P_\ell(\mu) . \tag{375}$$

Substituting μ_o from equation (375) into equation (372) gives

$$\overline{P}_1(E',\mu_o) = \sum_{\ell',\ell=0}^{\infty} \frac{2\ell+1}{2} \left(-\frac{1}{A}\right)^{\ell'} \overline{P}_\ell(E',\mu) \int_{-1}^{1} \left(\mu + \frac{1}{A}\right) P_{\ell'}(\mu) P_\ell(\mu) d\mu \, . \tag{376}$$

The recursion relation

$$(2\ell' + 1)\mu \, P_{\ell'}(\mu) = (\ell' + 1)P_{\ell'+1}(\mu) + \ell' P_{\ell'-1}(\mu) \tag{377}$$

can be introduced into equation (376) to obtain

$$\overline{P}_1(E',\mu_o) = \sum_{\ell,\ell'=0}^{\infty} \frac{(2\ell+1)}{2} \left(-\frac{1}{A}\right)^{\ell'} \overline{P}_\ell(E',\mu) \int_{-1}^{1} \left[\frac{\ell'+1}{2\ell'+1} P_{\ell'+1}(\mu) \right. \tag{378}$$

$$\left. + \frac{\ell'}{2\ell'+1} P_{\ell'-1}(\mu) + \frac{1}{A} P_{\ell'}(\mu) \right] P_\ell(\mu) d\mu \, .$$

Performing the indicated integration in equation (378) and employing the orthogonality relation

$$\int_{-1}^{1} P_m(\mu) P_n(\mu) d\mu = \frac{2}{2m+1} \delta mn$$

leads to

$$\overline{P}_1(E',\mu_o) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{A^{\ell+1}} \overline{P}_\ell(E',\mu) \left[\frac{\ell+2}{2\ell+3} - \frac{\ell}{2\ell-1} A^2 \right] . \tag{379}$$

Therefore, the P_2 approximation to $\overline{P}_1(E', \mu_o)$ is

$$\left[\overline{P}_1(E',\mu_o)\right]^{(2)} = \frac{2}{3A} + \overline{P}_1(E',\mu)\left[1 - \frac{3}{5A^2}\right] + \overline{P}_2(E',\mu)\left[\frac{4}{7A^3} - \frac{2}{3A}\right].$$

Average Square of the Cosine of the Scattering Angle in the Laboratory System.--The average value of $\mu_o^2 = \cos^2\psi$ is given by equation (328) as

$$\overline{\mu}_o^2 = \int_{-1}^{1} \mu_o^2 f(\mu)\,d\mu . \tag{381}$$

The P_2 approximation to $\overline{\mu}_o^2$ can be obtained by substituting the P_2 approximation to $f(\mu)$ from equation (28) and μ_o from equation (16) into equation (381). This procedure gives

$$\overline{\mu}_o^{2(2)} = \int_{-1}^{1} \frac{A^2\mu^2 + 2A\mu + 1}{A^2 + 2A\mu + 1}\left[\frac{1}{2} + \frac{3}{2}\overline{P}_1(E',\mu)\mu + \frac{5}{2}\overline{P}_2(E',\mu)\left\{\frac{3\mu^2-1}{2}\right\}\right]d\mu$$

$$= \frac{3}{4} - \frac{A^2}{4} + \frac{(A^2-1)^2}{8A}\ln\frac{A+1}{A-1}$$

$$+ \frac{3}{2}\overline{P}_1(E',\mu)\left[\frac{A^2+1}{2}\ln\frac{A+1}{A-1} - A\right] \tag{382}$$

$$+ \frac{5}{4}\overline{P}_2(E',\mu)\left[\left\{A + \frac{1}{16A^3}(A^2+1)^2(3A^2-1)(A^2-3)\right\}\ln\frac{A+1}{A-1}\right.$$

$$\left. - \frac{1}{8}\left(\frac{1}{A^2} - 1 - A^2 + A^4\right)\right] .$$

However, the desired theoretical input function is

$$\overline{P}_2(E',\mu_o) = \frac{1}{2}(3\overline{\mu}_o^2 - 1) \tag{383}$$

rather than $\overline{\mu}_o^2$. Combining equations (383) and 383) yields

$$\left[\overline{P}_2(E',\mu_o)\right]^{(2)} = \frac{1}{8}\left[5 - 3A^2 + \frac{3}{2A}(A^2-1)^2 \ln\frac{A+1}{A-1}\right]$$

$$+ \frac{9}{8}\overline{P}_1(E',\mu)\left[(A^2+1)\ln\frac{A+1}{A-1} - 2A\right] \tag{384}$$

$$+ \frac{15}{8}\overline{P}_2(E',\mu)\left[\left\{1 + \frac{1}{16A^4}(A^2+1)^2(3A^2-1)(A^2-3)\right\} A\ln\frac{A+1}{A-1}\right.$$

$$\left. - \frac{3}{8}\left(\frac{1}{A^2} - 1 - A^2 + A^4\right)\right] .$$

This relation can be put into a form which is more useful for computation purposes by expanding $\ln\frac{A+1}{A-1}$ in an infinite series of the form

$$\ln x = \sum_{n=1}^{\infty} \frac{2}{2n-1}\left(\frac{x-1}{x+1}\right)^{2n-1} . \tag{385}$$

Substituting

$$\ln\frac{A+1}{A-1} = \sum_{n=1}^{\infty} \frac{2}{2n-1}\left(\frac{1}{A}\right)^{2n-1} \tag{386}$$

into equation (384) and combining terms leads to

$$\left[\overline{P}_2(E',\mu_o)\right]^{(2)} = 3\sum_{n=1}^{\infty} \frac{1}{(2n+3)(4n^2-1)}\left(\frac{1}{A}\right)^{2n}$$

$$+ 9\overline{P}_1(E',\mu)\sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \left(\frac{1}{A} \right)^{2n-1} \tag{387}$$

$$+ 15\overline{P}_2(E',\mu) \sum_{n=0}^{\infty} \frac{4n^2 + 4n + 3}{(2n + 5)(4n^2 - 1)(4n^2 - 9)} \left(\frac{1}{A} \right)^{2n} .$$

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